Why3: Computational Real Numbers

MPRI Project Report

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Throughout this project, I installed and used the following solvers:

Solver	Version
Alt-Ergo	2.2.0
CVC4	1.6
Z3	4.8.4
CVC3	2.4.1
Eprover	2.2
Spass	3.7

Most of the assertions were proved with Alt-Ergo and CVC4 (less often with Z3, and even more rarely with CVC3, Eprover and Spass). As a macOS user, the installation of Z3 was problematic (its "counterexample" counterpart was the only one to be recognized by the Why3 IDE), so much so that I had choice but to modify my `.why3.conf` file by explicitly adding a block enforcing the use of Z3:

```
[prover]
command = "z3 -smt2 -T:%t sat.random_seed=42 nlsat.randomize=false smt.random_seed=42 %f"
command_steps = "z3 -smt2 sat.random_seed=42 nlsat.randomize=false smt.random_seed=42
    memory_max_alloc_count=%S %f"
driver = "z3_440"
editor = ""
in_place = false
interactive = false
name = "Z3"
shortcut = "z3"
version = "4.8.4"
```

2. Functions on Integers

Q1-4. Give an implementation of

`power2`, `shift_left` using `power2`

- `power2` and `shift_left` are straightforward: the only notable point is the **for** loop invariant in `power2`:

  ```
  let res = ref 1 in
  for i=0 to l-1 do invariant { !res = power 2 i }
    res *= 2
  done;
  !res
  ```

  which expresses the fact that the reference variable `res` stores the suitable power of 2 at each iteration, and trivially ensures that the postcondition holds:

  - at the last iteration:

    * `!res` contains 2^{l-1} at the beginning of the body loop
    * its value is then doubled, which results in `!res` being equal to 2^l
  - one exits the loop, and `!res` yielded at the end, whence satisfying the postcondition `result = power 2 l` of `power2`

`ediv_mod`, and `shift_right` using `ediv_mod`.

- given `ediv_mod` and `power2`, `shift_right` is easily defined as `let d, _ = ediv_mod z (power2 l)in d` and poses no difficulty.
- `ediv_mod` is slightly more tricky, but nothing to be afraid of: `d` and `r` are respectively the quotient and the rest of the well-known euclidean division of `x` by `y > 0`.

1. we first tackle the case where $x = \overbrace{|x|}^{\text{denoted by x\_abs}} \geq 0$: as it happens,

```
let x_abs = if x >= 0 then x else -x in
let d = ref 0 in
let r = ref x_abs in
while !r >= y do
  invariant { !r >= 0 && x_abs = !d * y + !r}
  variant { !r }
  incr d;
  r -= y
done;
```

   - the invariant $r \geq 0 \quad \wedge \quad \texttt{x\_abs} = dy + r$ is initially true, and remains so at each iteration of the loop as d (resp. r) is incremented (resp. decremented) by 1 (resp. y).
   - the **`while`** loop condition $r \geq y$ and the fact that $y > 0$ (precondition requirement of `ediv_mod`) justify the decreasing and well-founded variant `!r`
   - at the end the **`while`** loop:
     * $0 \leq \ r < y$
     * $\texttt{x\_abs} = dy + r$

     which provides a trivially correct implementation of the euclidean division, provided $x \geq \ 0$

2. otherwise, if $x < 0$, we reduce this to the previous case, by computing the corresponding d_abs and r_abs for $\texttt{x\_abs} = |x| = -x$
   - if $\texttt{r\_abs} = 0$: then $\texttt{x\_abs} = \texttt{d\_abs} \times y$, and $x = (-\texttt{d\_abs}) \times y$.

     One yields $d \overset{\text{def}}{=} -\texttt{d\_abs}, \quad r \overset{\text{def}}{=} 0$. This is easily discharged by CVC4 (we can even go as far as to add the extra assertion `assert { x = - !d * y }` to help the provers, but it shouldn't be necessary).
   - else if $\texttt{r\_abs} > 0$: then

$$\begin{cases} 0 \leq y - \texttt{r\_abs} < y \\ x = -\texttt{x\_abs} = -\texttt{d\_abs}\, y - \texttt{r\_abs} = (-\texttt{d\_abs} - 1)\, y + (y - \texttt{r\_abs}) \end{cases}$$

     Therefore, one yields $d \overset{\text{def}}{=} -\texttt{d\_abs} - 1, \quad r \overset{\text{def}}{=} y - \texttt{r\_abs}$.

This is discharged by CVC4 too, but we can add the assertion `assert { x = (- !d - 1)* y + y - !r && 0 <= y - !r < y }` to convince the provers.

Q5. Give an implementation of `isqrt`

When it comes to the sheer body of the function, as seen in class:

```
let function isqrt (n:int) : int
  requires { 0 <= n }
  ensures { result = floor (sqrt (from_int n)) }
  =
    let count = ref 0 in
    let sum = ref 1 in
    while !sum <= n do
      incr count;
      sum += 2 * !count + 1
    done;
    !count
```

However, proving the postcondition `result = floor (sqrt (from_int n))` turns out to be trickier than the one we saw in class (i.e. `sqr !count <= !n < sqr (!count + 1)`), in so far as all the specification pertaining to `floor` in the standard library is:

```
  function floor real : int

  axiom Floor_int :
    forall i:int. floor (from_int i) = i

  axiom Floor_down:
    forall x:real. from_int (floor x) <= x < from_int (Int.(+) (floor x) 1)

  axiom Floor_monotonic:
    forall x y:real. x <= y -> Int.(<=) (floor x) (floor y)
```

That is, the standard-library properties related to $\lfloor\bullet\rfloor$ on which the provers can rely are:

- $\lfloor\bullet\rfloor$ is increasing and left inverse of `from_int`
- and more importantly:

$$\forall n \in \mathbb{Z}, n = \lfloor x \rfloor \implies n \leq x < n+1 \qquad \circledast$$

On top of that, `sqrt` is only assumed to be increasing, and not strictly increasing.

As a result, we:

- *neither* have the converse of $\circledast$ (which is exactly the direction needed to prove the postcondition!)
- *nor* do we have the fact that $\sqrt{\bullet}$ is strictly increasing (which is problematic when dealing with strict inequalities).

So, which assertions where added to prove `isqrt`?

- concerning the **`while`** loop: nothing special, we proceed exactly as seen in class, apart from the extra variant: `variant {n - !sum}` which is easily seen to be strictly decreasing and well-founded.

- at the end of the loop:

$$0 \leq \texttt{count} \qquad \text{and} \qquad \texttt{count}^2 \leq n < \texttt{sum} = (\texttt{count} + 1)^2$$

therefore, due to $\sqrt{\bullet}$ being strictly increasing and $\texttt{count} \geq 0$:

$$\texttt{count} \leq \sqrt{n} < \texttt{count} + 1$$

and the converse of ⊛ would yield the expected postcondition.

But to convince the provers, based solely on the standard-library specification, we proceed as follows:

  - we first show that $\texttt{count} \leq \lfloor \sqrt{n} \rfloor$, which only resorts to $\lfloor \bullet \rfloor$ and $\sqrt{\bullet}$ being increasing and $\sqrt{\bullet}$ being a left inverse of $\bullet^2$ on $\mathbb{R}^+$ (axiom `Square_sqrt` of the standard library).
  - we then show the reverse inequality, that is: $\lfloor \sqrt{n} \rfloor < \texttt{count} + 1$ in a similar fashion. Except that this one is a bit trickier, as $\sqrt{\bullet}$ is not assumed to be strictly increasing, but we can get away with it by treating strict inequalities as being equivalent to non-strict ones *and* non-equalities.

3. Difficulty with Non-linear Arithmetic on Real Numbers

3.1 Power Function

Q6-12. Prove that

1. _B is positive
2. $\_\mathsf{B}\, n \times \_\mathsf{B}\, m = \_\mathsf{B}(n+m)$
3. $\_\mathsf{B}\, n \times \_\mathsf{B}(-n) = 1$
4. $0 \leq a \implies \sqrt{a \times \_\mathsf{B}(2n)} = \sqrt{a} \times \_\mathsf{B}\, n$
5. $0 \leq y \implies \_\mathsf{B}\, y = \texttt{from\_int}\ 4^y$
6. $y < 0 \implies \_\mathsf{B}\, y = \frac{1}{\texttt{from\_int}\ 4^{-y}}$
7. $0 \leq y \implies 2^{2y} = 4^y$

All theses lemmas but the 5th and the 6th ones are straightforwardly discharged:

- for the 5th one (`_B_spec_pos`): we lend a hand to the provers with the command `assert (pow (from_int 4) (from_int n)= from_int (power 4 n))`:

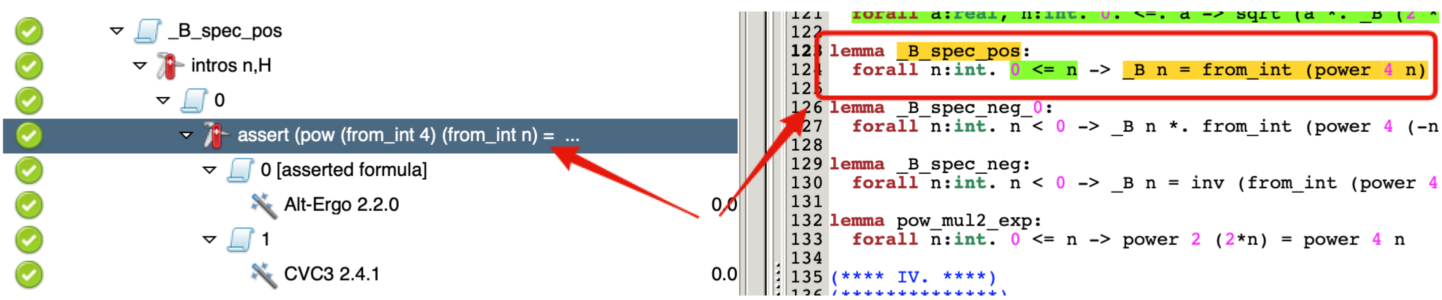

Figure 1: Why3 IDE: use of the `assert` command to prove `_B_spec_pos`

- for the 6th one (`_B_spec_neg`), we first prove an easily discharged (by Alt-Ergo) lemma:

```
lemma _B_spec_neg_0:
  forall n:int. n < 0 -> _B n *. from_int (power 4 (-n)) = 1.
```

  from which `_B_spec_neg` immediately ensues.

4. Computational Real Numbers

Q13. Could you find a reason why this definition is better than the other for automatic provers?

- When it comes to using to two inequalities rather the terser (and perhaps more elegant)

$$|x - p\,4^{-n}| < 4^{-n}$$

  the two-inequalities version has the advantage of not involving the absolute value `abs`, which would just be a burden when proving framing-related postconditions. Indeed, almost every time we would want to show a non-trivial framing (first needing to unfold `abs`), provers would eventually have to resort to the `Abs_le` lemma of the standard library, leading to unnecessary proof clutter.
- As for using `_B`: this fosters the use of the relevant lemmas proved in section **3.6** by the provers, bringing about more efficient proofs.

Q14. Prove these three functions

`round_z_over_4`

By dint of assertions, we show the two postconditions inequalities separately:

- $$\texttt{from\_int}\ (\underbrace{\texttt{shift\_right}\ \ (z+2)\,2}_{=(z+2)/\!/2^2}) \le (\texttt{from\_int}\ \ z+2) \times \texttt{\_B}(-1)$$

  where // stands for the euclidean division quotient, which directly stems from

$$4((z+2)\ /\!/\ 2^2) \le z+2 \qquad \text{(euclidean division)}$$

- Similarly (the `from_int` 's will be omitted from now on):

$$z - 2 < 4 \times \underbrace{\texttt{shift\_right}\ \ (z+2)\,2}_{=(z+2)/\!/2^2}$$

  due to

$$z - 2 < z + 2 - (\underbrace{(z+2) \mod 2^2}_{<4}) = 4((z+2)\ /\!/\ 2^2)$$

`compute_round` and `compute_add`

- For `compute_round`, assuming

$$(z_p - 2) \times \_\mathsf{B}(-(n+1)) < z \leq (z_p + 2) \times \_\mathsf{B}(-(n+1))$$

  we show that

$$(\underbrace{\texttt{shift\_right} \ \ (z_p+2)\,2}_{=(z_p+2)/\!/2^2} - 1) \times \_\mathsf{B}(-n) < z < ((z_p+2) /\!/ 2^2 + 1) \times \_\mathsf{B}(-n)$$

  by means of two assertions (one for each inequality). Indeed:

$$\begin{aligned}
((z_p+2) /\!/ 2^2 - 1) \times _\mathsf{B}(-n) &\leq \Big(\underbrace{\frac{z_p+2}{4} - 1}_{=\frac{z_p}{4}-\frac{1}{2}}\Big) \times _\mathsf{B}(-n) && \text{since } 4((z_p+2) /\!/ 2^2) \leq z_p + 2 \\
&= \frac{z_p - 2}{4} \times _\mathsf{B}(-n) \\
&= (z_p - 2) \times _\mathsf{B}(-(n+1)) \\
&< z \\
&\leq \frac{z_p + 2}{4} \times _\mathsf{B}(-n) \\
&= \Big(\frac{z_p - 2}{4} + 1\Big) \times _\mathsf{B}(-n) \\
&< ((z_p+2) /\!/ 2^2 + 1) \times _\mathsf{B}(-n) && \text{since } z_p - 2 < 4((z_p+2) /\!/ 2^2) \text{ as seen before}
\end{aligned}$$

- Given `compute_round`'s contract, `compute_add n x xp y yp` is straightforwardly defined as `compute_round n (x +. y)(xp + yp)`

4.2 Subtraction

Q15-16. Define and prove the functions `compute_neg`, `compute_sub` using `compute_neg` and `compute_add`

Those pose no difficulty:

- `compute_neg n x xp` is nothing more than `-xp`, as demonstrated by multiplying the framing of `x` by -1
- `compute_sub n x xp y yp` compute_adds `x` and the compute_neg'ed approximation of `y`, owing to `x` and `y` being provided at approximation $n+1$. A little help for the provers: asserting `assert { framing (-.y)yp' (n+1)}` just before yielding the result.

4.3 Conversion of Integer Constants

`compute_cst` is easy on paper, but is a bit thornier in Why3: we show the relevant inequalities in each case

- if $n < 0$:

- $(x \mathbin{//} 2^{-2n} - 1) \times \_\mathsf{B}(-n) < x$ stems from $(x \mathbin{//} 2^{-2n}) \times \_\mathsf{B}(-n) \leq x$ (by definition of the euclidean division) and $\_\mathsf{B}(-n) > 0$
- $x < (x \mathbin{//} 2^{-2n} + 1) \times \_\mathsf{B}(-n)$ comes from x being equal to $(x \mathbin{//} 2^{-2n}) \times \_\mathsf{B}(-n) + \underbrace{(x \mod \_\mathsf{B}(-n))}_{<\_\mathsf{B}(-n)}$

• if $n \geq 0$:

- $(x \times 2^{2n} - 1) \times \_\mathsf{B}(-n) = \underbrace{x \times 2^{2n} \times \_\mathsf{B}(-n)}_{=x} - \underbrace{\_\mathsf{B}(-n)}_{>0} < x$
- $x < x + \underbrace{\_\mathsf{B}(-n)}_{>0} = x \times 2^{2n} \times \_\mathsf{B}(-n) + \_\mathsf{B}(-n) = (x \times 2^{-2n} + 1) \times \_\mathsf{B}(-n)$

4.4 Square Root

Q17. Prove these two relations

It can be noted that, for all $n \in \mathbb{N}$:

$$(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = (n+1) - n = 1$$

$$\text{so that} \quad \sqrt{n+1} = \sqrt{n} + \underbrace{\frac{1}{\sqrt{n+1} + \sqrt{n}}}_{\text{denoted by } \texttt{\_sqrt\_incr}\, n}$$

$$\text{where} \quad 0 < \texttt{\_sqrt\_incr}\, n \leq 1$$

Based on this observation, we show two lemma functions

```
let lemma _sqrt_incr_spec (n:int) : unit
  requires { n >= 0 }
  ensures { sqrt (from_int (n+1)) = sqrt (from_int n) +. _sqrt_incr n }
  =
  (* [...] *); ()

let lemma _sqrt_incr_bounds (n:int) : unit
  requires { n >= 0 }
  ensures { 0. <. _sqrt_incr n <=. 1. }
  =
  (* [...] *); ()
```

that will come in handy in what follows.

Relation 1 (`sqrt_ceil_floor` lemma): $\lceil \sqrt{n+1} \rceil - 1 \leq \lfloor \sqrt{n} \rfloor$

The outline of the proof on paper is:

$$\begin{aligned}
\lceil\sqrt{n+1}\rceil - 1 < \lceil\sqrt{n+1}\rceil & \\
&= \lceil\sqrt{n} + \texttt{_sqrt_incr}\, n\rceil && \text{as } \sqrt{n+1} = \sqrt{n} + \texttt{_sqrt_incr}\, n \\
&\leq \lceil\underbrace{(\lfloor\sqrt{n}\rfloor + 1) + 1}_{\in\mathbb{Z}}\rceil && \text{since } \begin{cases}\sqrt{n} \leq \lfloor\sqrt{n}\rfloor + 1 \\ \texttt{_sqrt_incr}\, n \leq 1\end{cases} \quad \text{and } \lceil\bullet\rceil \text{ is increasing} \\
&= \underbrace{\lfloor\sqrt{n}\rfloor + 1}_{\text{denoted by } a} + 1
\end{aligned}$$

But we have actually more than that: $\lceil\sqrt{n+1}\rceil$ is *strictly lower* than $a+1$.

Indeed: if, by contradiction, we had $\lceil\sqrt{n+1}\rceil = a+1$, given that:

$$\sqrt{n} < \lfloor\sqrt{n}\rfloor + 1 = a = \lceil\sqrt{n+1}\rceil - 1 < \sqrt{n+1}$$

it would come that $n < a^2 < n+1$, which is absurd, since a^2 is an integer. So

$$\lceil\sqrt{n+1}\rceil - 1 < \lceil\sqrt{n+1}\rceil < a+1 = \lfloor\sqrt{n}\rfloor + 2$$

and as all these are integers, the result follows.

The reasoning by contradiction is carried out in Why3 in this way:

```
if ceil x = a+1 then (
    assert {  n-1 < a*a < n
           by (* [...] *) };
    absurd);
(* [...] *)
```

Relation 2 (`sqrt_floor_floor` lemma): $\lfloor\sqrt{n}\rfloor \leq \lfloor\sqrt{n-1}\rfloor + 1$

We proceed analogously, everything is similar:

$$\begin{aligned}
\lfloor\sqrt{n}\rfloor &= \lfloor\sqrt{n-1} + \texttt{_sqrt_incr}\, n\rfloor \\
&\leq \lfloor(\lfloor\sqrt{n-1}\rfloor + 1) + 1\rfloor \\
&= \underbrace{\lfloor\sqrt{n-1}\rfloor + 1}_{\text{denoted by } a} + 1
\end{aligned}$$

and $\lfloor\sqrt{n}\rfloor = a+1$ is impossible, as otherwise $\sqrt{n-1} < \lfloor\sqrt{n-1}\rfloor + 1 = a = \lfloor\sqrt{n}\rfloor - 1 < \sqrt{n}$, which would imply $n-1 < a^2 < n$.

Q18. Prove `compute_sqrt`

Assuming that

$$x \geq 0 \quad \text{and} \quad (x_p - 1) \times \_\mathsf{B}(-2n) < x < (x_p + 1) \times \_\mathsf{B}(-2n)$$

we show that

```
let compute_sqrt (n: int) (ghost x : real) (xp : int)
  = if xp <= 0 then 0 else isqrt xp
```

ensures that the `result` is an n-framing of $\sqrt{x}$.

- if $x_p \leq 0$, then:
$$-\_\mathsf{B}(-n) < 0 \leq \sqrt{x} < \sqrt{\underbrace{(x_p + 1)}_{=1} \times \_\mathsf{B}(-2n)} = \_\mathsf{B}(-n)$$

- if $x_p > 0$:

$$\sqrt{x} < \sqrt{x_p + 1} \times \_\mathsf{B}(-n) \leq \left\lceil \sqrt{x_p + 1} \right\rceil \times \_\mathsf{B}(-n) \overset{\text{Relation 1}}{\leq} \left(\left\lfloor \sqrt{x_p} \right\rfloor + 1 \right) \times \_\mathsf{B}(-n)$$
$$\sqrt{x} > \sqrt{x_p - 1} \times \_\mathsf{B}(-n) \geq \left\lfloor \sqrt{x_p - 1} \right\rfloor \times \_\mathsf{B}(-n) \overset{\text{Relation 2}}{\geq} \left(\underbrace{\left\lfloor \sqrt{x_p} \right\rfloor}_{= \texttt{isqrt } x_p} - 1 \right) \times \_\mathsf{B}(-n)$$

In Why3, we use the same trick as in `isqrt` to get around the fact that `sqrt` is not strictly increasing, by turning some strict inequalities into conjunctions of non-strict ones and non-equalities.

4.5 Compute

Q19-20. Define: `interp` that gives real interpretation of a term, and `wf_term` that checks that square root is adequately applied.

- `interp` is recursively defined in a forthright manner
- `wf_term` is defined as an inductive predicate. For the time being, the only non-trivial constructor case (that actually does check something, rather than inductively propagating) is `wf_sqrt: forall t:term. interp t >=. 0. -> wf_term t -> wf_term (Sqrt t)`, ensuring that `Sqrt` is exclusively applied to terms whose interpretation is non-negative.

Q21. define and prove the `compute` function

The first batch of the `compute` function is the following one:

```
let rec compute (t:term) (n:int) : int
  requires { wf_term t }
  ensures { framing (interp t) result n }
  variant { t }
  =
  match t with
    | Cst x -> compute_cst n x
    | Add t' t'' ->
        let xp = compute t' (n+1) in
        let yp = compute t'' (n+1) in
```

```
          compute_add n (interp t') xp (interp t'') yp
    | Neg t' -> compute_neg n (interp t') (compute t' n)
    | Sub t' t'' ->
        let xp = compute t' (n+1) in
        let yp = compute t'' (n+1) in
          compute_sub n (interp t') xp (interp t'') yp
    | Sqrt t' -> compute_sqrt n (interp t') (compute t' (2*n))
  end
```

It is defined by structural induction over the term `t`, which makes the `variant` trivially correct, and as all the contracts of the auxiliary `compute_***` functions were specially written to ensure the correction of this final `compute`, CVC4 discharges the proof obligations with no trouble.

5 Division

Q22. Prove these two properties

Notations: in what follows, we will denote by d (resp. d', resp. d'') and r (resp. r', resp. r'') the quotient and the rest of the euclidean division of a by b (resp. $b-1$, resp. $b+1$). In other words:

$$\begin{aligned} a &= db + r & 0 &\le r < b \\ a &= d'(b-1) + r' & 0 &\le r' < b-1 \\ a &= d''(b+1) + r'' & 0 &\le r'' < b+1 \end{aligned}$$

Property 1 (`dividend_incr`)

$$\begin{cases} 0 < a \\ 0 < b \\ d \overset{\text{def}}{=} a \,/\!/\, b < b \end{cases} \Longrightarrow \quad d'' \overset{\text{def}}{=} a \,/\!/\, (b+1) = \begin{cases} d-1 & \text{if } r \overset{\text{def}}{=} a \mod b < d \qquad \textbf{(P1.1)} \\ d & \text{else} \qquad \textbf{(P1.2)} \end{cases}$$

Assume $a, b > 0$ and $d \overset{\text{def}}{=} a \,/\!/\, b < b$.

- if $r \overset{\text{def}}{=} a \mod b < d$:

  Let us show that $d'' \overset{\text{def}}{=} a \,/\!/\, (b+1) = d - 1$.

  To do so, based on the lemma function suggested in the problem statement at the end of section 2 (which is easily proved by CVC4):

  ```
  let lemma euclid_uniq (x y q : int) : unit
    requires { y > 0 }
    requires { q * y <= x < q * y + y }
    ensures { ED.div x y = q }
    = ()
  ```

it suffices to show that

$$(d-1)(b+1) \leq a < d(b+1)$$

And indeed

- $(d-1)(b+1) = db + d - b - 1 \leq db + r = a$ since $d \leq b - 1 \leq b + r + 1$
- $a = db + r < db + b = d(b+1)$ as $r < b$

- if $r \geq d$:

  Let us show that $d'' = d$. Similarly:

$$d(b+1) \leq a < (d+1)(b+1)$$

in so far as

- $d(b+1) = db + d \leq db + r = a$
- $a = db + r < db + d + b + 1 = (d+1)(b+1)$ since $r < b$

Property 2 (`dividend_decr`)

$$\begin{cases} 0 < a \\ 1 < b \\ d \overset{\text{def}}{=} a \mathbin{/\!/} b < b - 1 \end{cases} \implies d' \overset{\text{def}}{=} a \mathbin{/\!/} (b-1) = \begin{cases} d+1 & \text{if } b-1-d < r \overset{\text{def}}{=} a \mod b \qquad \textbf{(P2.1)} \\ d & \text{else} \qquad \textbf{(P2.2)} \end{cases}$$

Assume $a > 0, b > 1$ and $d \overset{\text{def}}{=} a \mathbin{/\!/} b < b - 1$.

- if $b - 1 - d \leq r \overset{\text{def}}{=} a \mod b$:

  Let us show that $d' \overset{\text{def}}{=} a \mathbin{/\!/} (b-1) = d + 1$. Indeed:

$$(d+1)(b-1) \leq a < (d+2)(b-1)$$

because

- $(d+1)(b-1) = db + b - 1 - d \leq db + r = a$ due to the hypothesis
- $a = db + r < db - d + 2b - 2 = (d+2)(b-1)$ since $r < b + \underbrace{b-d-2}_{\geq 0}$

- if $b - 1 - d > r$:

  Let us show that $d' = d$. Similarly:

$$d(b-1) \leq a < (d+1)(b-1)$$

owing to the fact that

- $d(b-1) = db - d \leq db + r = a$ as $0 < a = db + r < (d+1)\overset{>0}{\widehat{b}}$ hence $d \geq 0$, and $-d \leq 0 \leq r$
- $a = db + r < db - d + b - 1 = (d+1)(b-1)$ because of the hypothesis

The two lemma functions `dividend_incr` and `dividend_decr` closely follow the proof sketches above in the Why3 implementation.

Q23. Prove the function `inv_simple_simple`

We first prove two routine lemmas (`inv_decreasing`: the fact that `inv` is decreasing over $\mathbb{R}_+^*$ and `_B_inv`: $\forall n,\ \_\mathrm{B}\, n = \frac{1}{\_\mathrm{B}(-n)}$) that are subsequently used in `inv_simple_simple`.

```
let inv_simple_simple (ghost x:real) (p:int) (n:int)
    requires { framing x p (n+1) }
    requires { 0 ≤n }
    requires { 1. ≤. x }
    ensures { framing (inv x) result n }
    =
    let k = n + 1 in
    let d,r = ediv_mod (power2 (2*(n+k))) p in
    if 2*r ≤p then d
    else d+1
```

As far as I am concerned, `inv_simple_simple` was the most nettlesome function, and maybe the most confusing one too at first glance, for the following reason: as pointed out in the problem statement, we can (and we will) prove that

$$d = a \mathbin{/\!/} b \leq b - 1 - a \mathbin{/\!/} b$$

which ensures that the conditions **P1.1** and **P2.1** cannot happen at the same time, that is: **P1.1** $\implies$ **P2.2** and **P2.1** $\implies$ **P1.2**. From there, it is tempting to try to show (in each branch of `inv_simple_simple`'s **`if`** statement) one the first conditions of one property (**P1.1** or **P2.1**), since, as it happens, the second condition of the other property is obtained for free. But that's a misleading track! We will instead focus on the second conditions of one property (i.e. either **P1.2** or **P2.2**), disregarding the other property altogether (by just settling with the *coarsest upper/lower bound* we get from both of its conditions).

Let's delve into it in more details. Similarly to before, we set

$$\begin{aligned}
(d, r) &= (4^{n+k} \mathbin{/\!/} p, 4^{n+k} \mod p) \\
(d', r') &= (4^{n+k} \mathbin{/\!/} (p-1), 4^{n+k} \mod (p-1)) \\
(d'', r'') &= (4^{n+k} \mathbin{/\!/} (p+1), 4^{n+k} \mod (p+1))
\end{aligned}$$

- Before entering the **`if`** statement: we prove a handful of useful assertions
  - $4 \leq 4^k \leq p$ and $4^n \leq \frac{p}{4}$

    since $1 \leq x < (p+1)4^{-k}$, so $4^k < p+1$, whence $4^k \leq p$. On top of that: $k = n+1$ (thus $4^n \leq \frac{p}{4}$) and $k \geq 1$ (hence $p \geq 4$).
  - then, as we have the precondition `framing x p (n+1)` (i.e. `framing x p k`):

$$\frac{4^k}{p+1} < \frac{1}{x} < \frac{4^k}{p-1}$$

therefore

$$d'' \leq \frac{\overbrace{4^{n+k}}^{(p+1)d''+r''}}{p+1} < \frac{4^n}{x} < \frac{\overbrace{4^{n+k}}^{(p-1)d'+r'}}{p-1} \leq d'+1$$

- $d \leq \frac{p-1}{2}$. Indeed:

$$\begin{aligned}
& d = \frac{4^{n+k}-r}{p} \leq \frac{p-1}{2} \\
\iff\quad & 4^{n+k} - r \leq \frac{p(p-1)}{2} \\
\Longleftarrow\quad & 4^{n+k} \leq \frac{p(p-1)}{2} \\
\Longleftarrow\quad & \frac{p^2}{4} \leq \frac{p(p-1)}{2} \\
\Longleftarrow\quad & \frac{p}{2} \leq p-1 \\
\Longleftarrow\quad & 2 \leq p && \text{which is true as } p \geq 4
\end{aligned}$$

- last but not least (before entering the `if`): the *coarsest bounds* we hinted at earlier:
  * due to **P2**: $d' \leq d+1$
  * due to **P1**: $d-1 \leq d''$
- Inside the `if` statement:
  - **if** $2r \leq p$**:**
    * Let us show that $\boxed{r+1 \leq p-1-d}$:

$$\begin{aligned}
& r+d+1 < p \\
\iff\quad & rp + \underbrace{dp}_{=4^{n+k}-r \leq \frac{p^2}{4}-r} + p < p^2 \\
\Longleftarrow\quad & \underbrace{r}_{\leq \frac{p}{2}}(p-1) + \frac{p^2}{4} + p < p^2 \\
\Longleftarrow\quad & 2p(p-1) + p^2 + 4p < 4p^2 \\
\iff\quad & 0 < p^2 - 2p = p(p-2) && \text{which is true as } p \geq 4
\end{aligned}$$

    * Thus, by **P2.2**, $d' = d$. And consequently:

$$d - 1 \overset{\text{coarse bound}}{\leq} d'' < \frac{4^n}{x} < d'+1 = d+1$$

  - **if** $2r > p$**:**

- It comes that $\boxed{r \geq d}$, since $2r \geq\ p+1 \geq p-1 \geq\ 2d$
- Thus, by **P1.2**, $d'' = d$. And consequently:

$$(d+1)-1 = d'' < \frac{4^n}{x} < d'+1 \overset{\text{coarse bound}}{\leq} (d+1)+1$$

Q24. Prove the function `inv_simple`

`inv_simple` take advantage of the fact that $1 \leq x \times \_\mathrm{B}\, m$ to resort to `inv_simple_simple`. We are given a $(n+1+2m)$-framing of x:

$$\begin{aligned} &(p-1)\,\_\mathrm{B}(-(n+1+2m)) < x < (p+1)\,\_\mathrm{B}(-(n+1+2m)) \\ \text{hence } &(p-1)\,\_\mathrm{B}(-(n+1+m)) < x \times \_\mathrm{B}\, m < (p+1)\,\_\mathrm{B}(-(n+1+m)) \end{aligned}$$

and as $1 \leq x \times \_\mathrm{B}\, m$, `res = inv_simple_simple (x *. _B m)p (n+m)` provides a $(n+m)$-framing of $x \times \_\mathrm{B}\, m$:

$$\begin{aligned} &(\mathtt{res}-1)\,\_\mathrm{B}(-(n+m)) < x \times \_\mathrm{B}\, m < (\mathtt{res}+1)\,\_\mathrm{B}(-(n+m)) \\ \text{thus } &(\mathtt{res}-1)\,\_\mathrm{B}(-n) < \underbrace{x \times \_\mathrm{B}\, m \times \_\mathrm{B}(-m)}_{=x} < (\mathtt{res}+1)\,\_\mathrm{B}(-n) \end{aligned}$$

and the result follows.

Q25. extend the type `term`

We add

- the `| Inv t' -> inv (interp t')` case in `interp`
- the `| wf_inv: forall t:term. interp t <> 0. -> wf_term t -> wf_term (Inv t)` case in `wf_term`

Q26-27. prove the correction and termination of both functions

- When it comes to the correction:

  nothing really fancier than before: the only new case is `Inv t'`, and `msd` (which is called only there in `compute`) yields an `m` such that $|\mathtt{interp}\ t| > \_\mathrm{B}(-m)$ (such an `m` always exists provided $\mathtt{interp}\ t \neq 0$, which is what we assume).

  `msd` recursively calls itself until $|c| \geq\ 2$ (where c is the integer approximating t), thus straightforwardly ensuring the correction of the algorithm.

  In `compute`, the case where the sign is negative is easily treated, similarly to `compute_neg`, by taking the opposite.

- The termination is a bit more involved because of `msd`:
  - when `compute t n` is called:
    - either `t` is structurally smaller
    - either `t` remains the same and `compute` has been called inside `msd`

    which hints at the fact that an adequate variant would follow a lexicographic order, with the size of `t` as first component (where `size` is defined as expected).
  - `msd` stops recursively calling itself as soon as $|c| > 1$.
    - if $\texttt{interp} \;\; t > 0$:

      then if $4^n(\texttt{interp} \;\; t) > 2$, i.e. $n > \log_4 \frac{2}{\texttt{interp} \;\; t} = \log_4 \frac{2}{|\texttt{interp} \;\; t|}$, it follows that $c > 4^n(\texttt{interp} \;\; t) - 1 > 1$
    - if $\texttt{interp} \;\; t < 0$:

      then if $4^n(\texttt{interp} \;\; t) < -2$, i.e. $n > \log_4 \frac{-2}{\texttt{interp} \;\; t} = \log_4 \frac{2}{|\texttt{interp} \;\; t|}$, it follows that $c < 4^n(\texttt{interp} \;\; t) + 1 < -1$

    and each time `msd` is recursively called, n is incremented (and it is originally set at 0).

  So a good variant is $(\texttt{size} \;\; t, \left\lceil \log_4 \frac{2}{|\texttt{interp} \;\; t|} \right\rceil - n)$ for the lexicographic order, which we can routinely check in Why3 by asserting what was outlined before and adding axiom about the `log` being increasing as suggested at the beginning of the problem statement.

Bonus

Here is a counter-example: with

- $x \stackrel{\text{def}}{=} -0.6161$
- $n = 2$

It comes that $\texttt{msd} \;\; (x) = 1$ with $x_0 = 0, x_1 = -2, ..., x_5 = -630$.

Let's run the proposed algorithm on this instance to compute, say, $\overline{1/x_n}$.

- $n > -\texttt{msd} \;\; (x) = -1$
- as $k = n + 2\texttt{msd} \;\; (x) + 1 = 5$ and $x_5 = -630 \leq 1$: it comes that

$$\overline{1/x_2} = \left\lfloor \frac{\mathsf{\_B}(k+n)}{x_k} \right\rfloor = -27$$

However:

$$(\overline{1/x_n} + 1) \times \mathsf{\_B}(-n) = \frac{-27+1}{16} \simeq -1.625 < \frac{1}{x} \simeq -1.623$$

so the framing is not correct.

Conclusion

I didn't find this project particularly easy (especially as I am not keen on real numbers computation usually), but it definitely was a good foray into Why3. The most difficult part was `inv_simple_simple`, due to the fact that I got bogged down in a misleading track (as explained before) by misinterpreting a cue in the problem statement.

With some of my friends, I have jotted down a handful of suggestions about axioms that I think could be good adjuncts to the standard library, and a few features that may enhance the user experience of the Why3 IDE. I will enclose them in a forthcoming email.