ARPE September presentation: Generalised Species of Structures

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Ecole Normale Supérieure Paris-Saclay

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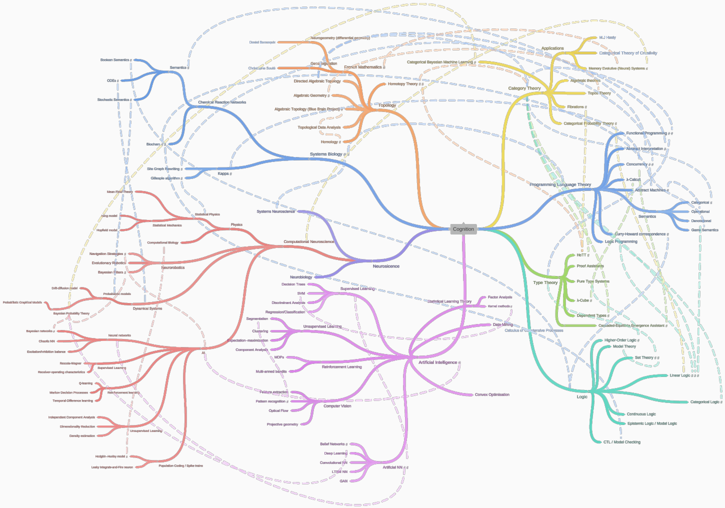

Mind map of explored fields over my years at the ENS

Author & Institution

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Project

Generalised Species of Structures [10, 11, 8, 1]

Project will involve

- presheaf categories (cf. M1 at Oxford [16])

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- monoidal and higher categories (cf. M2 at Macquarie in Sydney [15])
- homotopy type theory (cf. L3 at Nottingham [14])

Context

Species of structures

Generalised species of structures

2008: "The cartesian closed bicategory of generalised species of structures" by Fiore, Gambino, Hyland, and Winskel [10]
Generalisation of

- **1981:** Joyal's **species of structures** [13]

Motto: Data type/computation structures from a combinatorial perspective, and vice versa.

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  - extra rules to produce *derivatives* of proofs

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Generalised species of structures

Also related to:

- ∞-categories via **polynomial functors** [17]

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  - resource calculi [18]
- **Homotopy Type Theory** [2]: their calculus can be mimicked therein

Joyal's species: definition

Combinatorial species of structures P

A functor

$$P\colon \underset{\substack{\uparrow \\ \text{groupoid of finite sets}}}{\mathsf{B}} \longrightarrow \mathsf{Set}$$

Equivalently: P given by a family of symmetric group actions

$$\_[=]\colon P[n] \times \mathfrak{S}_n \longrightarrow P[n]$$

such that

$$p[\mathrm{id}] = p \qquad p[\sigma][\tau] = p[\sigma \cdot \tau] \qquad \forall p \in P[n],\ \sigma, \tau \in \mathfrak{S}_n$$

Joyal's species: intuition

Intuition: For a species $P\colon \mathbf{B} \longrightarrow \mathbf{Set}$

- $P(U)$ = structures of type P parameterised by the set of tokens U
- action of P = abstract rule of transport of structures (structural equivalence)

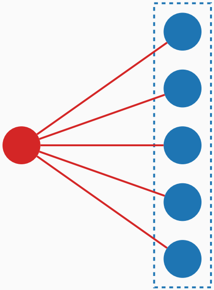


Figure 1: Schematic representation of a species

Joyal's species: generating series

Generating series of $P\colon \mathsf{B} \longrightarrow \mathsf{Set}$:

$$P(x) = \sum_{n \geq 0} |P[n]| \, \frac{x^n}{n!}$$

Equality of species:

Natural isomorphism (pointwise bijection + well-behaved with transport)

Arithmetic on generating functions $(+, \times, \circ, \partial) \longleftrightarrow$ Combinatorial calculus of species

Joyal's species: examples

Examples

- Endofunctions:
  - $\mathrm{End}(U) \;:=\; \mathrm{Hom}_{\mathsf{Set}}\,(U,U) \qquad \forall U \in \mathsf{B}$
  - $\mathrm{End}(\sigma)(f) \;:=\; \sigma\, f\, \sigma^{-1} \qquad \forall f \in \mathrm{End}(U,U),\ \sigma \in \mathrm{Hom}_{\mathsf{B}}\,(U,V)$
- Underlying set species $\mathcal{U}\colon \mathsf{B} \hookrightarrow \mathsf{Set}$
- $X^n \;:=\; \mathrm{Hom}_{\mathsf{B}}\,(n,-)$
- Terminal species: $\exp(X) \;:=\; U \longmapsto \{U\}$
- Initial species: $0 \;:=\; U \longmapsto \emptyset$

Joyal's species: addition and multiplication

- Addition:

$$(P+Q)(U) \; := \; P(U)+Q(U)$$

- Multiplication: Day's tensor product

$$P \cdot Q \; := \; \int^{U_1, U_2 \in \mathbf{B}} P(U_1) \times Q(U_2) \times \mathrm{Hom}_{\mathbf{B}}\left(U_1 + U_2, -\right)$$

$\leadsto$ In $[\mathbf{B}, \mathbf{Set}]$:

$$(P \cdot Q)(U) = \sum_{U_1 \sqcup U_2 = U} P(U_1) \times Q(U_2)$$

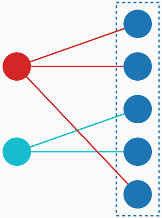


Figure 2: Multiplication of species

Joyal's species: differentiation

- Differentiation:

$$(\mathrm{d}/\mathrm{d}x)\, P = P(- + x)$$

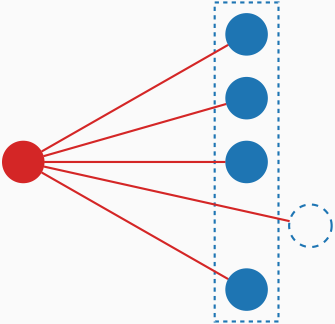


Figure 3: Differentiation of species

Joyal's species: composition

- Composition:

$$(Q \circ P)(U) \ := \int^{T \in \mathbf{B}} Q(T) \times (\underbrace{P \cdot \dots \cdot P}_{|T| \text{ fois}})(U)$$

$\rightsquigarrow$ In $[\mathbf{B}, \mathbf{Set}]$:

$$(Q \circ P)(U) = \sum_{\mathcal{U} \in \mathrm{Part}\, U} Q(\mathcal{U}) \times \prod_{u \in \mathcal{U}} P(u)$$

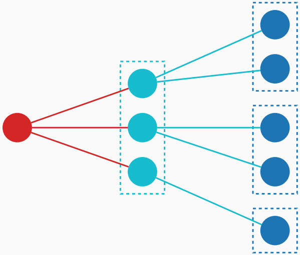


Figure 4: Composition of species

Joyal's species: composition

Analytic endofunctors on Set

Species $\longleftrightarrow$ Coefficients of analytic endofunctors on **Set**

In this respect:

Composition of species = Composition of corresponding functors

Studied article: “Generalised Species of Structures: Cartesian Closed and Differential Structure”

Generalised species: definition

Marcelo's draft article: "Generalised Species of Structures: Cartesian Closed and Differential Structure" [8]

For $\mathbb{A}, \mathbb{B}$ small categories:

Generalised $(\mathbb{A}, \mathbb{B})$-species of structures P

A functor

$$\underset{\text{free symmetric strict monoidal completion}}{P\colon\ \underset{\uparrow}{!\mathbb{A}}} \longrightarrow \overset{\text{presheaves on } \mathbb{B}}{\overset{\downarrow}{\widehat{\mathbb{B}}}}$$

Generalised species: examples

Examples

- Joyal's species are $(1, 1)$-species
- But also: Permutationals [13], partitionals, presheaves, the Yoneda embedding, ...

Graphical calculus: $P\colon\, !A \longrightarrow \widehat{B}$ depicted pictorially as

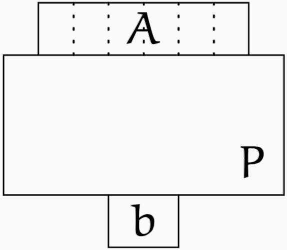
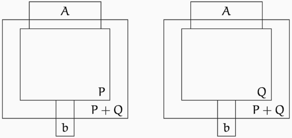


Figure 5: Schematic representation of a generalised species

Generalised species: addition and multiplication

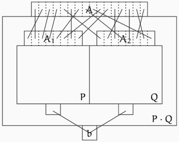
- Addition:

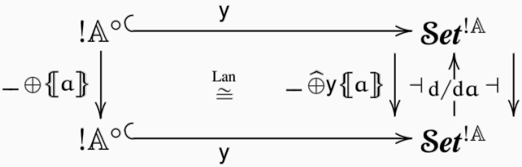
$$(P+Q)(A)(b) \;:=\; P(A)(b) + Q(A)(b)$$



- Multiplication:

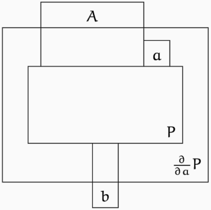
$$(P \cdot Q)(A)(b) \;:=\; \int^{A_1, A_2 \in \mathbf{B}} P(A_1)(b) \times Q(U_2)(b) \times \mathrm{Hom}_{!\mathbb{A}}\,(A_1 \oplus A_2, A)$$

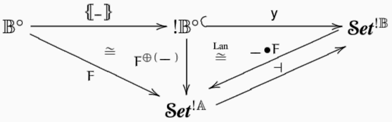




- Differentiation:

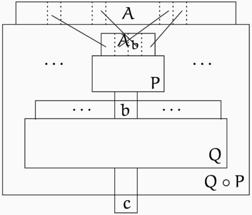
$$(\partial/\partial a)\, P(A)(b) = P(A \oplus [\![a]\!])(b)$$





- Composition:

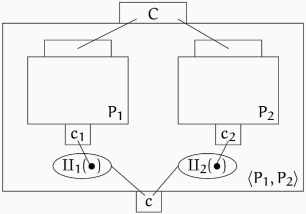
$$(Q \circ P)(A)(c) = \int^{B \in !\mathbb{B}} Q(B)(c) \times \overbrace{P^{\#}(A)(B)}^{:= \int^{X \in (!\mathbb{A})^{|B|}} \left(\prod_{k \in |B|} P(X_k)(B_k) \right) \times \mathrm{Hom}_{!\mathbb{A}} \left(\bigoplus_{k \in |B|} X_k, A \right)}$$



Generalised species: Cartesian Closed Structure (products)

- Pairing of $P_i \colon !\mathbb{C} \longrightarrow \widehat{\mathbb{C}_i}$:

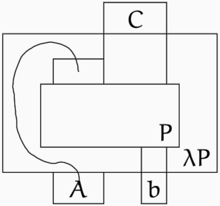
$$\langle P_i \rangle_{i \in I}(C)(c) = \sum_{i \in I} \int^{z \in \mathbb{C}_i} P_i(C)(z) \times \mathrm{Hom}_{\prod_{i \in I} \mathbb{C}_i}\left(c, \coprod_i z\right)$$



Generalised species: Cartesian Closed Structure (exponentiation)

- Exponentiation:

$$\begin{array}{c} !\coprod_1 C \oplus !\coprod_2 A \\ \downarrow \\ (\lambda_{\mathbb{A}} P)(C)(A, b = P(C \otimes A)(b) \end{array} \qquad \forall C \in !\mathbb{C}, A \in !\mathbb{A}, b \in \mathbb{B}$$



Tedious proofs

For example, to show the basic associativity and unit laws:

Proposition 2.3 *For* $\mathrm{P} : \mathbb{A} \mathrel{\mapstochar\!\!\rightarrow} \mathbb{B}$, $\mathrm{Q} : \mathbb{B} \mathrel{\mapstochar\!\!\rightarrow} \mathbb{C}$, *and* $\mathrm{R} : \mathbb{C} \mathrel{\mapstochar\!\!\rightarrow} \mathbb{D}$, *we have canonical natural coherent isomorphisms as follow*

$$\boxed{\begin{array}{c} \mathrm{I}_{\mathbb{B}} \circ \mathrm{P} \cong \mathrm{P} \cong \mathrm{P} \circ \mathrm{I}_{\mathbb{A}} \\ (\mathrm{R} \circ \mathrm{Q}) \circ \mathrm{P} \cong \mathrm{R} \circ (\mathrm{Q} \circ \mathrm{P}) \end{array}} \tag{10}$$

establishing the unit laws of identities and the associativity of composition.

Proof:

$$\begin{aligned} &(\mathrm{I}_{\mathbb{B}} \circ \mathrm{P})(A)(b) \\ &\quad = \int^{B \in !\mathbb{B}} !\mathbb{B}\left[\{\!\!\{b\}\!\!\}, B\right] \times \int^{A_{b'} \in !\mathbb{A}\ (b' \in B)} \left(\textstyle\prod_{b' \in B} \mathrm{P}(A_{b'})(B_{@b'})\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{b' \in B} A_{b'}, A\right] \\ &\quad \cong \int^{b' \in \mathbb{B}} \mathbb{B}[b, b'] \times \int^{A' \in !\mathbb{A}} \mathrm{P}(A')(b') \times \mathbb{A}[A', A] \\ &\quad \cong \int^{b' \in \mathbb{B}} \mathbb{B}[b, b'] \times \mathrm{P}(A)(b') \\ &\quad \cong \mathrm{P}(A)(b) \end{aligned} \tag{11}$$

$$\begin{aligned} &(\mathrm{P} \circ \mathrm{I}_{\mathbb{A}})(A)(b) \\ &\quad = \int^{A' \in !\mathbb{A}} \mathrm{P}(A')(b) \times \int^{X_a \in !\mathbb{A}\ (a \in A')} !\mathbb{A}\left[\{\!\!\{A'_{@a}\}\!\!\}, X_a\right] \times !\mathbb{A}\left[\textstyle\bigotimes_{a \in A'} X_a, A\right] \\ &\quad \cong \int^{A' \in !\mathbb{A}} \mathrm{P}(A')(b) \times !\mathbb{A}\left[\textstyle\bigotimes_{a \in A'} \{\!\!\{A'_{@a}\}\!\!\}, A\right] \\ &\quad \cong \mathrm{P}(A)(b) \end{aligned} \tag{12}$$

$$\begin{aligned} &((\mathrm{R} \circ \mathrm{Q}) \circ \mathrm{P})(A)(d) \\ &= \int^{B \in !\mathbb{B}} (\mathrm{R} \circ \mathrm{Q})(B)(d) \times \int^{A_b \in !\mathbb{A}\ (b \in B)} \left(\textstyle\prod_{b \in B} \mathrm{P}(A_b)(B_{@b})\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{b \in B} A_b, A\right] \\ &= \int^{B \in !\mathbb{B}} \left(\int^{C \in !\mathbb{C}} \mathrm{R}(C)(d) \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\textstyle\prod_{c \in C} \mathrm{Q}(B_c)(C_{@c})\right) \times !\mathbb{B}\left[\textstyle\bigotimes_{c \in C} B_c, B\right]\right) \\ &\qquad\quad \times \int^{A_b \in !\mathbb{A}\ (b \in B)} \left(\textstyle\prod_{b \in B} \mathrm{P}(A_b)(B_{@b})\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{b \in B} A_b, A\right] \\ &\cong \int^{C \in !\mathbb{C}} \mathrm{R}(C)(d) \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\textstyle\prod_{c \in C} \mathrm{Q}(B_c)(C_{@c})\right) \\ &\quad \times \int^{A_b \in !\mathbb{A}\ (b \in \bigotimes_{c \in C} B_c)} \left(\textstyle\prod_{b \in (\bigotimes_{c \in C} B_c)} \mathrm{P}(A_b)\left((\bigotimes_{c \in C} B_c)_{@b}\right)\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{b \in (\bigotimes_{c \in C} B_c)} A_b, A\right] \\ &\cong \int^{C \in !\mathbb{C}} \mathrm{R}(C)(d) \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\textstyle\prod_{c \in C} \mathrm{Q}(B_c)(C_{@c})\right) \\ &\quad \times \int^{A_{b,c} \in !\mathbb{A}\ (c \in C, b \in B_c)} \left(\textstyle\prod_{c \in C, b \in B_c} \mathrm{P}(A_{b,c})\left((B_c)_{@b}\right)\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{c \in C} \bigotimes_{b \in B_c} A_{b,c}, A\right] \end{aligned} \tag{13}$$

Tedious proofs

$$
\begin{aligned}
&(R \circ (Q \circ P))(A)(d) \\
&= \int^{C \in !\mathbb{B}} R(C)(d) \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \left(\textstyle\prod_{c \in C} (Q \circ P)(A_c)(C_{@c})\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{c \in C} A_c, A\right] \\
&= \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\textstyle\prod_{c \in C} \int^{B \in !\mathbb{B}} Q(B)(C_{@c}) \\
&\qquad\qquad\qquad \times \int^{X_b \in !\mathbb{A}\ (b \in B)} \left(\textstyle\prod_{b \in B} P(X_b)(B_{@b})\right) \\
&\qquad\qquad\qquad\qquad \times !\mathbb{A}\left[\textstyle\bigotimes_{b \in B} X_b, A_c\right] \Big) \\
&\qquad\qquad \times !\mathbb{A}\left[\textstyle\bigotimes_{c \in C} A_c, A\right] \\
&\cong \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\textstyle\prod_{c \in C} Q(B_c)(C_{@c})\right) \\
&\qquad\qquad\qquad \times \Big(\textstyle\prod_{c \in C} \int^{X_b \in !\mathbb{A}\ (b \in B_c)} \left(\textstyle\prod_{b \in B_c} P(X_b)((B_c)_{@b})\right) \\
&\qquad\qquad\qquad\qquad \times !\mathbb{A}\left[\textstyle\bigotimes_{b \in B_c} X_b, A_c\right] \Big)\Big) \\
&\qquad\qquad \times !\mathbb{A}\left[\textstyle\bigotimes_{c \in C} A_c, A\right] \\
&\cong \int^{C \in !\mathbb{B}} R(C)(d) \qquad\qquad (14) \\
&\qquad \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\textstyle\prod_{c \in C} Q(B_c)(C_{@c})\right) \\
&\qquad\qquad\qquad \times \Big(\int^{X_{b,c} \in !\mathbb{A}\ (c \in C, b \in B_c)} \left(\textstyle\prod_{c \in C} \prod_{b \in B_c} P(X_{b,c})((B_c)_{@b})\right) \\
&\qquad\qquad\qquad\qquad \times \textstyle\prod_{c \in C} !\mathbb{A}\left[\textstyle\bigotimes_{b \in B_c} X_{b,c}, A_c\right] \Big)\Big) \\
&\qquad\qquad \times !\mathbb{A}\left[\textstyle\bigotimes_{c \in C} A_c, A\right] \\
&\cong \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\textstyle\prod_{c \in C} Q(B_c)(C_{@c})\right) \\
&\qquad\qquad \times \int^{X_{b,c} \in !\mathbb{A}\ (c \in C, b \in B_c)} \left(\textstyle\prod_{c \in C} \prod_{b \in B_c} P(X_{b,c})((B_c)_{@b})\right) \\
&\qquad\qquad\qquad \times !\mathbb{A}\left[\textstyle\bigotimes_{c \in C} \bigotimes_{b \in B_c} X_{b,c}, A\right]
\end{aligned}
$$

Overall results

Results

- composition $\longrightarrow$ generalized species of structures form a bicategory
- addition and multiplication $\longrightarrow$ commutative rig structure
- pairing/projection, abstraction/evaluation, and differentiation $\longrightarrow$ cartesian closed and linear structure

What’s next?

ARPE Project

1. Revising, finishing off and publishing the categorical theory of generalised species of structure [8]
2. Characterizing the exponentiable para-toposes mentioned in Marcelo's unpublished joint work with André Joyal [9] related to generalised species of structure $\longrightarrow$ further developments in connection to higher-dimensional category theory [7] and resource calculi [18]
3. Mimicing the calculus of generalized species in Homotopy Type Theory (HoTT) [2]
4. Connecting and investigating this from the ∞-categorial viewpoint via polynomial functors [17]
5. Free symmetric strict monoidal completion = symmetric Fock-space construction. Operators of creation/annihilation of particles in corresponding quantum systems + commutation. $\longrightarrow$ Feynman diagrams in the context of generalised species.

Conclusion

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