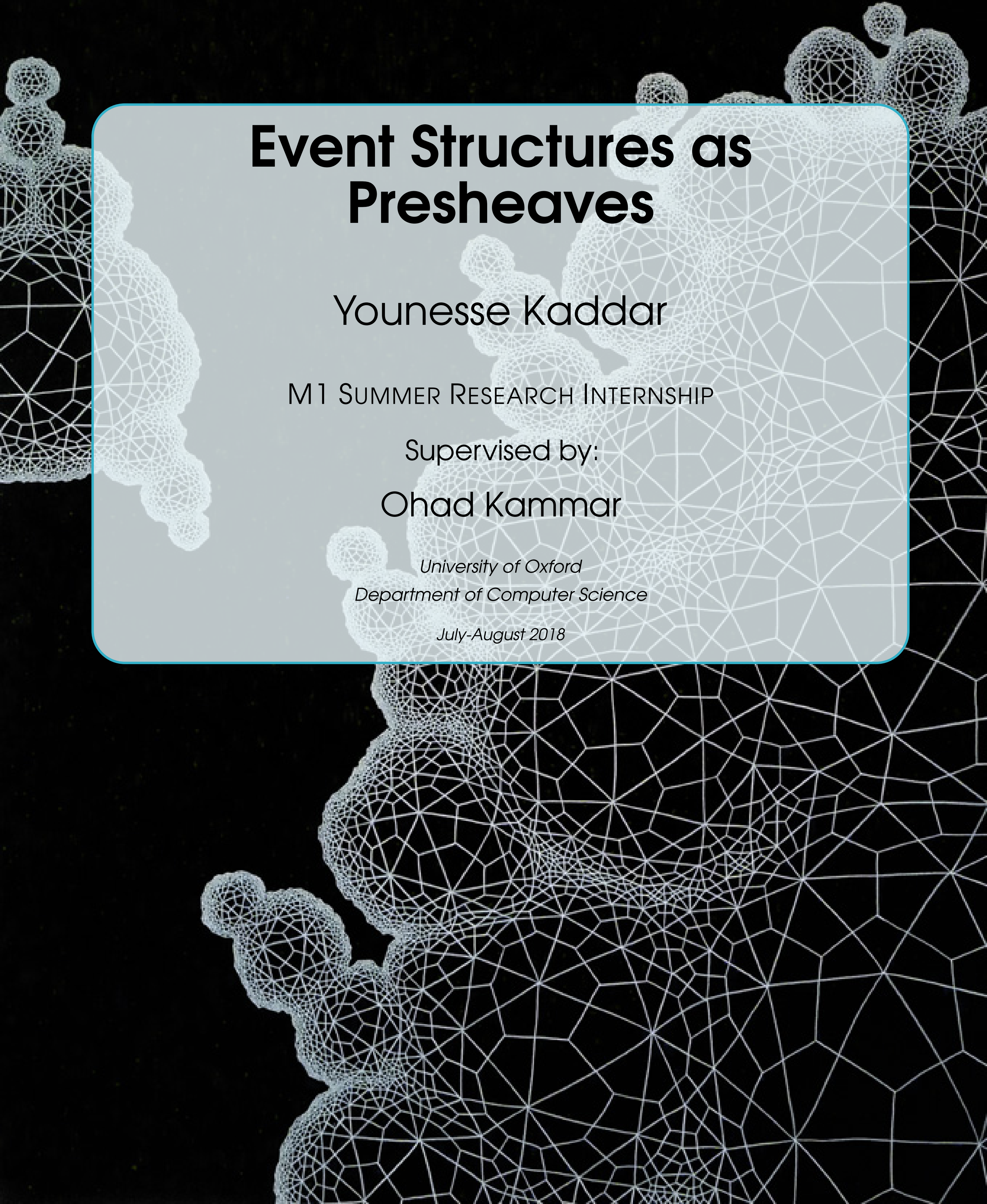

Event Structures as Presheaves

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M1 SUMMER RESEARCH INTERNSHIP

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1. Overview

General context

Glynn Winskel's event structures are a well-established model of concurrency: they can be thought of as a set of events that can be causally dependent on one another (causality/dependency relation) and/or exclude each other (conflict relation). An event structure can be equally regarded as a transition system, which can be thus studied from an operational point of view, and as a particular Scott domain (a prime algebraic one), which enables us to use all the machinery of domain theory. As such, event structures are really operationally informative in respect to analysing concurrent processes.

Problem statement

However, this concreteness turns out to be a poisoned chalice when it comes to studying concurrency from a categorical standpoint. For example, effects – which are phenomena happening in the background during the process execution – are crucial with regards to concurrent programs (due to side-effects and shared states, among others), but the theory of algebraic effects relies on monad which makes it hard to make it compatible with the operational facet of event structures. This is the reason why we are interested in describing event structures as a presheaf category: the goal of this internship is to investigate this matter.

Contribution

This was a short internship, I did not have time to delve into the problem nearly as much as I would have liked to. What I did was essentially folklore – except for the extension of natural transformations precomposed by a dense functor, which led to an alternative proof of the full- and faithfulness of the nerve functor – but I did my best to grasp the material at hand and do the proofs that were not specified in the papers. My work was divided into two parts:

1. first, understanding how event structures relate to Scott domains and prove basic properties thereof (most of which are in the appendix)
2. second, learn about various categorical concepts (among others: extranatural transformations, Grothendieck construction, ends/coends, co-Yoneda lemma, Kan extensions, density, coreflections, . . .), especially about presheaf categories and how they arise as free cocompletion of simpler categories (in particular: I tried to figure out what presheaves have to do with freely adding colimits), which eventually led up to the nerve realisation paradigm at work when embedding event structures into presheaves over (non-empty) finite partial orders of events.

Arguments in favour of what has been done

Seeing event structures as presheaves is a way to cross the gap that sets apart the operational aspect of transition systems from the categorical semantics viewpoint, which should in turn pave the way for studying them through the lens of the theory of algebraic effects. As a matter of fact, some presheaf models of concurrency have been studied over the past few years and are raising interest in the programming language theory community, in so far as bringing effects and concurrency together would be the next step toward designing richer and richer models and languages taking advantage of both theories.

Results and prospects

This internship was really enlightening, in that I studied concurrent computation through three different angles: the event structure, the domain theoretic and the categorical point of view. It was also the opportunity to deepen my knowledge and practical skills in category theory, and above all to work on the various problems I tackled in a team environment: Ohad Kammar, Mathieu Huot, Dario Stein, Paul Blain Lévy, Sean Moss, and Jesse Sigal have provided me with valuable support and advice throughout my stay. I volunteered at FLoC 2018 too, and it was an awesome experience!

Of course, this is just the beginning: I would really like to keep working with Ohad on this matter, the next stage would be to go a step further toward the theory of algebraic effects, by studying monads over event-structures with symmetry and rigid maps using algebraic operations and equations.

Acknowledgements

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I've had a wonderful stay at Oxford, Sam and Ohad's team members welcomed me so warmly that I felt at home right away!

Prerequisites

We assume basic familiarity with category theory (categories, functors, natural transformations, comma categories, co/slices, co/limits, adjoints, Yoneda lemma, . . .) and domain theory (Scott domains, Scott continuous functions, . . .), even though the most important notions that are assumed will often be recalled (see the reminders about comma categories A.2.3 and various other categorical notions B.1). Good introductions to category theory are [Lan98] and [Rie16] for example.

Notations			
$f[x]$	direct image of the set $x \subseteq A$ under the function $f : A \to B$	$F \downarrow G$	comma category of $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$
$\bigvee x$	least upper bound (lub) of $x \subseteq P$	Δ_a	Constant functor at a
$\bigwedge x$	greatest lower bound (glb) of $x \subseteq P$	$\cong$	Isomorphism
$\underline{E}$	Carrier set	$\simeq$	Equivalence

Conventions (though not systematic): General categories will be denoted by $\mathbf{C}, \mathbf{D}, \ldots$, small categories by $\mathbb{C}, \mathbb{D}, \ldots$, objects by $C, c, D, d, \ldots$.

Abbreviation: ”iff” will be the abbreviation of ”if and only if”.

2. Category of Event structures: $\mathcal{E}$

2.1 Event Structures

To begin with, let's revisit a historical example: in his 1986 book *The Society of Mind* [Min86], Marvin Minsky ventured that:

> [...] you can build a mind from many little parts, each mindless by itself.

He called *mental agents* the small processes that, according to him, would make up the human mind, each of which, very limited in its own right, could only perform a very simple and specific task. On the whole, all the agents would add up and interact in a complex manner to bring about what we call *intelligence* (this phenomenon is referred to as *society of mind* by him).

Minsky's favourite example (simplified)

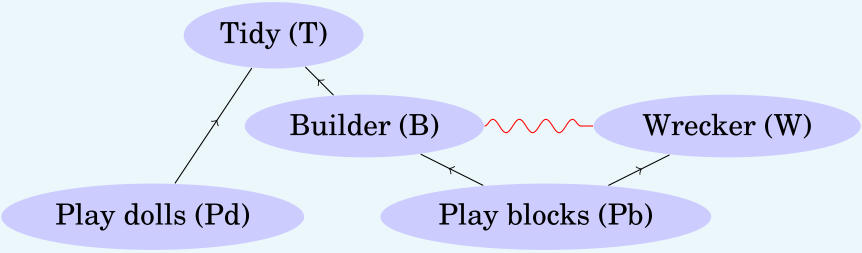
Let's study mental agents on a toy (no pun intended) example. Imagine a child who has the luxury to choose to play with dolls or (inclusive) Lego building blocks. The child's mental agent responsible for the former (resp. the latter) behaviour will be called Play dolls – abbreviated Pd (resp. Play blocks – Pb). Depending on the child's mood, let's say that there are two ways to play with the building blocks:

- either our child feels like keeping quiet and building a tower, in which case Play blocks calls a sub-process Builder (B)
- or he may just want to smash everything, in which case Play blocks initiates the agent Wrecker (W): he does not feel like building anything, let alone tidying afterwards! (the agent W may be thought of as being in conflict with B and the tidying one: Tidy (T)).

Our child is lazy: he tries to avoid cleaning up as much as possible, until he can no longer wriggle out of it without being scolded by his mother (*i.e.* when dolls *and* blocks are in a mess). As a result, he tidies his room – resorting to the mental agent Tidy – only when he played with his dolls *and* his blocks before.
The aforementioned mental agents form a simple example of *event structure*, summarised in the figure below: causality/dependency (resp. conflict) is denoted by black arrows (resp. red squiggly edges).

NB

- As W is in conflict with B and B begets T, one considers that W is bound to be in conflict with T as well, so the conflict link between T and W is implicit.
- T being dependent on Pd and B means that Pd *and* B are prerequisites for T to happen.



Event structure of the child's mental agents

Vocabulary 2.1 — A well-founded poset $E := \langle \underline{E}, \leq_E \rangle$ is a poset such that for all $e \in \underline{E}$, $\downarrow e := \{e' \in \underline{E} \mid e' \leq_E e\}$ (*downward closure* of e) is finite.

Definition 2.1.1 — An event structure $E := \langle \underline{E}, \leq_E, \mathrm{Con}_E \rangle$ is given by:

- a *well-founded* poset $(\underline{E}, \leq_E)$ of *events* (where $\leq$ is thought of as the *causal dependency relation*: $e', e'' \leq e \in \underline{E}$ means that e can occur only if e' and e'' have occurred before)
- a *consistency relation* $\mathrm{Con}_E \subseteq \mathcal{P}_{\mathrm{fin}}(E)$ such that:
    - for all $e \in \underline{E}$, $\{e\} \in \mathrm{Con}_E$;
    - Con_E is closed under subsets: $X \subseteq Y \in \mathrm{Con}_E \implies X \in \mathrm{Con}_E$; and
    - *augmentation property*: if $x \in \mathrm{Con}_E$ and $e' \leq_E e \in x$ then $x \cup \{e'\} \in \mathrm{Con}_E$.

The consistency relation Con_E is to be thought of as specifying which events are *not* incompatible with one another (and thus may happen at the same time). The augmentation property says that if e' is a cause for e to happen, adding that cause to any $x \in \mathrm{Con}_E$ containing e won't induce any conflict.

Example 2.1 In our playing child example, all the subsets of $\underline{E} := \{\mathrm{Pb}, \mathrm{Pd}, \mathrm{B}, \mathrm{T}, \mathrm{W}\}$ are in Con_E *except* those that contain W and B, or W and T. ■

Note that being in the consistency relation is restricted to finite subsets only; which brings us to the definition of being consistent, "extending" this, in a way, to infinite subsets:

Vocabulary 2.2 — Consistent subsets: A subset $x \subseteq \underline{E}$ is *consistent* when $\mathcal{P}_{\mathrm{fin}}(x) \subseteq \mathrm{Con}_E$.

NB $\underline{E} \supseteq x$ is consistent $\iff \begin{cases} x \in \mathrm{Con}_E & x \text{ is finite} \\ \mathcal{P}_{\mathrm{fin}}(x) \subseteq \mathrm{Con}_E & \text{otherwise} \end{cases}$

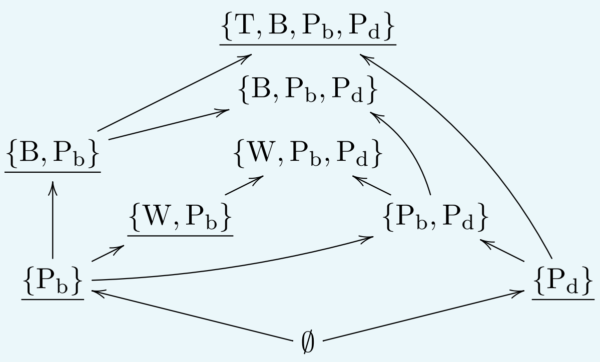
Another important notion is that of *down-closed subsets*:

Vocabulary 2.3 — Down-closed subsets: A subset $x \subseteq \underline{E}$ is *down-closed* when $\downarrow x := \bigcup_{e \in x} \downarrow e \subseteq x$

Down-closed subsets x are such that for all $e \in x$, x contains all the events that must have happened for e to occur. Combining the previous points, we're getting to the notion of *configurations*:

Definition 2.1.2 — The set of configurations is defined as $\mathcal{C}E := \{x \subseteq \underline{E} \mid x \text{ consistent and down-closed}\}$ We denote the set of finite configurations by $\mathcal{C}^o E$.

Example 2.2 — Playing child: poset of configurations (for the inclusion order).



■

A configuration corresponds to the event history of a partial run of a given computational process. As such, it contains all the events that entailed the point at which the partial run is considered, and incorporate no incompatible events. More than that: consistent subsets are precisely subsets of configurations:

Lemma — A.1.3 & A.1.4. A (finite) subset of events is consistent iff its downward closure is a (finite) configuration iff it occurs as a subset of a (finite) configuration.

2.2 Rigid maps

Morphisms (called *maps*) *of event structures* can be thought of as ”synchronising” two event structures, or as ”simulating”, so to say, the behaviour of one with the other:

Definition 2.2.1 — A map of event structures $f : E \to F$ is a function $f : \underline{E} \to \underline{F}$ that is

- *configuration preserving*: if $x \in \mathcal{C}^o E$, then $f[x] \in \mathcal{C}^o F$; and
- *locally injective*: for $x \in \mathcal{C}^o E$, if $e_1, e_2 \in x$ and $f(e_1) = f(e_2) \in F$, then $e_1 = e_2$.

A rigid map of event structures is a map which is also *monotone*: if $e_1 \leq_E e_2$, then $f(e_1) \leq_F f(e_2)$.

The local injectivity emphasises the fact that events are ”atomic”: an event in F cannot come from distinct consistent events in E.

Notation 2.1. *We denote by $\mathcal{E}$ the category of event structures and rigid maps.*

The rigidness condition is better understood with the following property:

Proposition — A.2.3. Every map $f : E \to F$ is *causality reflecting* on configurations:

$$\forall e_1, e_2 \in x \in \mathcal{C}^o E, \quad fe_1 \leq_F fe_2 \Longrightarrow e_1 \leq_E e_2$$

This means that if $x \in \mathcal{C}^o E$ and the map $f : E \longrightarrow F$ is in addition rigid, $f[x]$ and x are isomorphic as partial orders. Rigid maps have another key property: they reflect configurations:

Proposition — A.2.5. Every rigid map $f : E \to F$ is *configuration reflecting* on configurations:

$$\forall x' \subseteq x \in \mathcal{C}^o E,\ f[x'] \in \mathcal{C}^o F \Longrightarrow x' \in \mathcal{C}^o E$$

The previous proposition leads to this characterisation:

Lemma — A.2.6. A map $f : E \to F$ is rigid iff

$$\forall x \in \mathcal{C}^o E, \forall y \in \mathcal{C}^o F, \quad y \subseteq f[x] \Longrightarrow \exists x' \in \mathcal{C}^o E,\ x' \subseteq x \wedge fx' = y$$

Therefore, all the configurations of E can be seen as configurations of F, *i.e.* F can be regarded as ”extending” E. We end this section with a simple yet paramount example of event structures:

Example 2.3 — Well-founded posets. If P is a well-founded poset, $\langle \underline{P}, \leq_P, \mathcal{P}_{\text{fin}}(P) \rangle$ is an event structure (Lemma A.2.8). ■

Notation 2.2 (Category of paths). *$\mathbb{P}$ is the category whose **objects** are finite posets (seen as event structures) and **morphisms** are rigid maps. $I : \mathbb{P} \hookrightarrow \mathcal{E}$ is its full embedding in $\mathcal{E}$.*

3. Event Structures as Domains

Scott domains are special kinds of posets whose elements are thought of as pieces of partial information regarding a computation ($x \leq y$ meaning that the information carried by y ”extends” the one of x). They are extensively used in denotational semantics, thereby providing a mathematical meaning to programs. We will briefly give an account of particular Scott domains: finitary prime algebraic domain, and show how they are related to event structures.

3.1 Category of finitary prime algebraic domains: $\mathscr{P}$

The classic notion of *directed* sets is replaced by a slightly more general one: *finitely bounded* sets.

Vocabulary 3.1 — A subset $X \subseteq \underline{D}$ of a poset D is finitely bounded iff for all $Y \in \mathcal{P}_{\text{fin}}(\underline{X})$, y has an upper bound in D. **A poset D is consistently complete** iff every finitely bounded subset $X \subseteq \underline{D}$ has a least upper bound (lub) $\bigvee X$ in D.

Let D be is a consistently complete poset.

NB

- the condition « y has an upper bound in X » for directed sets has been replaced by « y has an upper bound in D ».
- if $\underline{D} \neq \emptyset$ then D has a least element: $\bigvee \emptyset$ (Lemma A.3.2)
- every non-empty subset $\emptyset \neq X \subseteq \underline{D}$ has a greatest lower bound (Lemma A.3.3)

The notion of *compact element* in Scott domains is substituted by *complete primes*:

Definition 3.1.1 — An element $p \in \underline{D}$ is a complete prime when for all $X \subseteq \underline{D}$ bounded, we have:

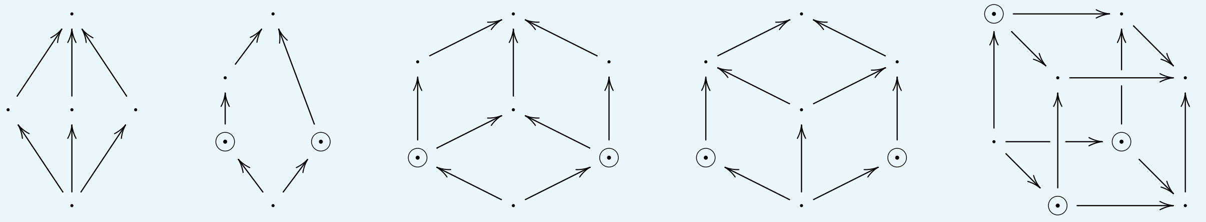
$$p \leq \bigvee X \implies \exists d \in X;\ p \leq d$$

NB In $\mathbb{N}$ ordered by divisibility, complete primes correspond to actual prime powers.

Vocabulary 3.2 — Prime downward-closure of an element $d \in \underline{D}$: $\downarrow^{\mathrm{p}} d := \{p \leq d \mid p \text{ complete prime}\}$

Definition 3.1.2 — A prime algebraic domain is a consistently complete poset D that satisfies the *prime algebraicity property*: for every $d \in \underline{D},\ d = \bigvee \downarrow^{\mathrm{p}} d$.

Example 3.1 — Posets and their complete primes (circled).



NB Only the last poset is a prime algebraic domain.

Example 3.2 Let's look at other examples and non-examples:

- $\mathbb{I} := [0,1]$ is *not* a prime algebraic domain (Proposition A.3.5).
- the ordinal $\omega + 1 := \{0 \leq 1 \leq 2 \leq \cdots \leq \omega\}$ is a prime algebraic domain (Proposition A.3.6).
- $(\omega+1)^2$ (the poset of pairs from $\omega + 1$ with the component-wise order) is a prime algebraic domain (Proposition A.3.7).

■

Moreover, as event structures are well-founded, we will need to restrict ourselves to prime algebraic domains that are said to be *finitary*:

Vocabulary 3.3 — A prime algebraic domain is said to be finitary if $\downarrow^{\mathrm{p}} p$ is finite for every complete prime p.

Finally, prime algebraic domain morphisms are none other than slightly generalised Scott continuous functions, that not only preserve the least upper bounds of directed sets, but also those of finitely bounded ones:

Definition 3.1.3 — A map of prime algebraic domains $f : C \to D$ is a monotone function such that for every finitely bounded $X \subseteq \underline{C}$ we have $\bigvee f[X] = f \bigvee X$.

Notation 3.1. *$\mathscr{P}$ denotes the category of finitary prime algebraic domains and their maps.*

3.2 Event structures as Finitary prime algebraic domains

The poset of configurations of an event structure determine a finitary prime algebraic domain:

Lemma — A.4.1. Let E be an event structure. The poset $\langle \mathcal{C}E, \subseteq \rangle$ is a prime algebraic domain, whose complete primes are the configurations $\{\downarrow e\}_{e \in \underline{E}}$

Example 3.3 In Example 2.2, the complete primes of the prime algebraic domain of configurations are underlined. ■

Reciprocally, from a finitary prime algebraic domain, one gets an event structure whose carrier set is the complete primes:

Lemma — A.4.2. Let D be a finitary prime algebraic domain, we have an event structure $\mathfrak{P}_D$ given by:

$$\underline{\mathfrak{P}_D} := \{p \in \underline{D} \mid p \text{ complete prime}\} \qquad p_1 \leq_{\mathfrak{P}_D} p_2 \iff p_1 \leq_D p_2 \qquad x \in \mathrm{Con}_{\mathfrak{P}_D} \iff x \text{ bounded in } \underline{D}$$

Lastly, going back and forth between these two construction yields the isomorphisms:

Lemma — A.4.3. In $\mathcal{E}$, every event structure $E \in \mathcal{E}$ satisfies:

$$E \simeq \mathfrak{P}_{\mathcal{C}E}$$

Lemma — A.4.4. In $\mathscr{P}$, every prime algebraic domain D satisfies:

$$D \simeq \langle \mathcal{C}\mathfrak{P}_D, \subseteq \rangle$$

4. Event Structures as Presheaves

4.1 Presheaves as cocompletion

Let $\mathbb{C}$ be a small category and $\mathbf{y}_{\mathbb{C}}$ its Yoneda embedding. The goal of this section is to show the following theorem and understand it intuitively:

Theorem 1 — The functor $\mathbf{y}_{\mathbb{C}}$ is the free cocompletion of $\mathbb{C}$.

For every cocomplete category $\mathbf{D}$ and functor $F : \mathbb{C} \to \mathbf{D}$ there is a unique (up to isomorphism) cocontinuous functor $\hat{F} : \widehat{\mathbb{C}} \to \mathbf{D}$ making the evident diagram commute up to natural isomorphism:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] \arrow[dr, phantom, "\cong"] & \mathbf{D} \\
\widehat{\mathbb{C}} \arrow[ur, dotted, "\hat{F}"'] &
\end{tikzcd}$$

A puzzling question may arise when considering the free cocompletion $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ of $\mathbb{C}$:

> “What on earth does freely adding all its colimits to $\mathbb{C}$ have to do with presheaves? 4.1.1”

4.1.1 Category of elements

A first approach to answer question 4.1.1 would be to investigate what happens when one adds, for each diagram in $\mathbb{C}$, its formal colimit, up until the resulting category $\widetilde{\mathbb{C}}$ ends up being cocomplete.

Example 4.1 — Adding a formal coproduct. For example, consider the simplest non-trivial example of colimit: the coproduct. Suppose that we have two distinguished objects $A, B \in \mathbb{C}$ and we want to define their formal coproduct – that will be denoted by Ω – in the category[1] $\widetilde{\mathbb{C}}$, so that the following universal property holds: for every functor $F : \mathbb{C} \longrightarrow \mathbf{D}$ such that $F(A), F(B)$ have a coproduct in $\mathbf{D}$, there exists a unique (up to isomorphism) functor $\hat{F} : \widetilde{\mathbb{C}} \to \mathbf{D}$ such that the evident triangle commutes:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook] & \mathbf{D} \\
\widetilde{\mathbb{C}} \arrow[ur, dotted, "\hat{F}"'] &
\end{tikzcd}$$

and $\hat{F}(\Omega) \cong F(A) + F(B)$

Then, defining $\Omega \in \widetilde{\mathbb{C}}$ amounts to describe, for all $C \in \mathbb{C}$, the morphisms:

- *going out of* Ω: this one is a cakewalk, we just use the universal property of the coproduct:

$$\mathrm{Hom}_{\widetilde{\mathbb{C}}}(\Omega, C) := \mathrm{Hom}_{\mathbb{C}}(A, C) \times \mathrm{Hom}_{\mathbb{C}}(B, C)$$

- *going into* Ω: By universal property of the coproduct, we do have a map

$$\mathrm{Hom}_{\widetilde{\mathbb{C}}}(C, A) + \mathrm{Hom}_{\widetilde{\mathbb{C}}}(C, B) \longrightarrow \mathrm{Hom}_{\widetilde{\mathbb{C}}}(C, \Omega)$$

Unfortunately, it may not be bijective at all in general (having a map into the coproduct is not tantamount to having a map into A and a map into B)! But here's the thing: if you want the previous universal property, you're bound to set:

$$\mathrm{Hom}_{\widetilde{\mathbb{C}}}\left(C, \Omega\right) := \mathrm{Hom}_{\mathbb{C}}\left(C, A\right) + \mathrm{Hom}_{\mathbb{C}}\left(C, B\right)$$

■

---

[1] $\mathbb{C}$ is a full subcategory of $\widetilde{\mathbb{C}}$ and $|\widetilde{\mathbb{C}}| := |\mathbb{C}| \cup \{\Omega\}$

Now, coming back to the general case of freely adding (in $\widehat{\mathbb{C}}$) the colimit Ω_D of any diagram[1] $D : I \longrightarrow \mathbb{C}$, we may be tempted to generalise from the previous example by setting, for all $C \in \mathbb{C}$:

$$\underbrace{\mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\Omega_D, C\right) := \lim_{i \in I} \mathrm{Hom}_{\mathbb{C}}\left(D_i, C\right)}_{\substack{\uparrow \\ \text{enforcing the continuity of } \mathbf{y}_{\mathbb{C}}(C) \text{ for all } C \in \mathbb{C}}} \qquad \text{and} \qquad \underbrace{\mathrm{Hom}_{\widehat{\mathbb{C}}}\left(C, \Omega_D\right) := \operatorname*{colim}_{i \in I} \mathrm{Hom}_{\mathbb{C}}\left(C, D_i\right)}_{\substack{\uparrow \\ \text{making all } C \in \mathbb{C} \text{ small}}}$$

But this approach is too naive: for instance, the two following diagrams are different (since their index categories are), but they ought to have the same colimit:

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}$$

So we don't want to add two distinct formal colimits (as we would if we were to add all the Ω_D for every diagram D) for them! How is it even possible to keep track of all the diagrams that should have the same colimit, in such a way that we would add theirs only once? This is where presheaves come into play! Indeed, each presheaf can be associated to a diagram in $\mathbb{C} \cong \mathbf{y}_{\mathbb{C}}\mathbb{C}$

Indeed, consider for instance the following diagram in $\mathbb{C} :=$

$$\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \arrow[dl, bend left, "h"] \\
C &
\end{tikzcd} :$$

$$\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] \arrow[dr, bend right=15, "f"] & B \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B
\end{tikzcd}$$

To go about adding its colimit P in $\widehat{\mathbb{C}}$, the trick is to rely on the isomorphism $\mathbb{C} \cong \mathbf{y}_{\mathbb{C}}[\mathbb{C}]$, so as to see the diagram in $\mathbf{y}_{\mathbb{C}}[\mathbb{C}] \subseteq [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] = \widehat{\mathbb{C}}$. Now, consider the colimiting cocone with summit P over the resulting diagram:

$$\begin{tikzcd}
& P & \\
\mathbf{y}_{\mathbb{C}}A \arrow[ur, bend left, "\varphi_1^A"] \arrow[rr, shift left, "\mathbf{y}_{\mathbb{C}}g"] \arrow[rr, shift right, "\mathbf{y}_{\mathbb{C}}f"'] \arrow[drr, bend right=15, "\mathbf{y}_{\mathbb{C}}f"'] & & \mathbf{y}_{\mathbb{C}}B \arrow[ul, bend right, "\varphi_1^B"'] \\
\mathbf{y}_{\mathbb{C}}A \arrow[uur, bend left=60, "\varphi_2^A"] \arrow[rr, bend right=10, "\mathbf{y}_{\mathbb{C}}g"'] & & \mathbf{y}_{\mathbb{C}}B \arrow[uul, bend right=60, "\varphi_1^B"']
\end{tikzcd}$$

As it happens, we are in front of the comma category $\mathbf{y}_{\mathbb{C}} \downarrow P$. And it turns out that it is isomorphic to the category of elements of P:

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[1] I is called the *index category* of the functor D

Lemma — B.1.7. If $Q \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$:

$$\int Q \cong \mathbf{y}_{\mathbb{C}} \downarrow Q$$

As a result, we have an explicit description of P: in the colimiting cocone,

- each natural transformation $\mathbf{y}_{\mathbb{C}} X \xrightarrow{\phi_i^X} P$ (where $X \in \{A, B\}, i \in \{1, 2\}$) corresponds to an element $x_i \in P(X)$
- for each morphism $$\begin{tikzcd} & P & \\ \mathbf{y}_{\mathbb{C}} X \arrow[ur, "\phi_i^X"] \arrow[rr, "\mathbf{y}_{\mathbb{C}} h"'] & & \mathbf{y}_{\mathbb{C}} Y \arrow[ul, "\phi_j^Y"'] \end{tikzcd}$$ (where $h \in \{f, g\}$), $Pf(y_j) = x_i$

And all the sets $\{P(C)\}_{C \in \mathbb{C}}$ and the action of P on the $\mathbb{C}$-morphisms are obtained in this way, by isomorphism with the category of elements of P (which unfolds everything there is to know about P: its action on sets and morphisms). So, in our example, P is given by:

$$P(A) = \{a_1, a_2\} \qquad P(B) = \{b_1, b_2\} \qquad P(C) = \emptyset$$

$$Pf = \begin{cases} P(B) \longrightarrow P(A) \\ b_i \mapsto a_1 \qquad \forall i \in \{1, 2\} \end{cases} \qquad Pg = \begin{cases} P(B) \longrightarrow P(A) \\ b_i \mapsto a_i \qquad \forall i \in \{1, 2\} \end{cases} \qquad Ph : P(C) = \emptyset \overset{\substack{\text{initial map in } \mathbf{Set} \\ \downarrow}}{\hookrightarrow} P(B)$$

So a presheaf has been associated to our original diagram, acting as its colimit. But there is a catch: everything goes well in the above example, but it might not in general! The highlighted sentence above Lemma B.1.7 is the fallacy of the argument: for a $\mathbb{C}$-diagram $D : I \longrightarrow \mathbb{C}$ whose colimit is denoted by $P_D \in \widehat{\mathbb{C}}$, it may not be the case in general that

Property 4.1.1

$$\mathbf{y}_{\mathbb{C}} D \downarrow P_D \cong \overbrace{\mathbf{y}_{\mathbb{C}} \downarrow P_D}^{\cong \int P_D}$$

But it can be shown that every diagram D is "equivalent" – in a sense that is made precise in the appendix B.1.1 – to a diagram D' that has this property. As such, the freely added colimit of D in $\widehat{\mathbb{C}}$ will be taken to be the presheaf $P_{D'}$. Any diagram $D : I \longrightarrow \mathbb{C}$ is "equivalent" to the diagram $\int(\operatorname{colim} \mathbf{y}_{\mathbb{C}} D) \xrightarrow{U} \mathbb{C}$, where U is the evident forgetful functor. Thus, it appears that $P_D \cong \operatorname{colim} \left(\underbrace{\int \underbrace{(\operatorname{colim} \mathbf{y}_{\mathbb{C}} D)}_{P_D}}_{\cong \mathbf{y}_{\mathbb{C}} \downarrow P} \xrightarrow{U} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}} \right)$. In general:

Theorem — B.1.15 - Every presheaf is a canonical colimit of representables. For all $P \in \widehat{\mathbb{C}}$,

$$P \cong \operatorname{colim} \left(\mathbf{y}_{\mathbb{C}} \downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}} \right)$$

A different take on the matter would be through the lens of coends: for every presheaf $P \in \widehat{\mathbb{C}}$, $P \cong \int^{c} Pc \times \mathbf{y}_{\mathbb{C}} c$ (this is referred to as the *co-Yoneda lemma*, see the appendix B.1.2 for more details).

4.1.2 Kan Extensions

Kan extensions are very expressive universal constructions that enable us to extend functors along one another. The Kan extension of a functor F can be thought of as the best approximation of F

extending its domain to a larger category. Their ubiquity throughout mathematics led MacLane to state in [Lan98]:

> The notion of Kan extensions subsumes all the other fundamental concepts of category theory. [p.248]

The approach resorting to Kan extensions will enable us to see the problem at a higher level of generality, from which Theorem 4.1 will ensue, rather than tackling the issue hands-on.

Definition 4.1.1 — A left Kan extension of a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ along a functor $K : \mathbf{C} \longrightarrow \widetilde{\mathbf{C}}$ is a functor $\mathrm{Lan}_K(F) : \widetilde{\mathbf{C}} \longrightarrow \mathbf{D}$ and a natural transformation $\eta : F \longrightarrow \mathrm{Lan}_K(F) \circ K$ (called the *unit*) which is an initial arrow from $F \in [\mathbf{C}, \mathbf{D}]$ to $- \circ K : [\widetilde{\mathbf{C}}, \mathbf{D}] \longrightarrow [\mathbf{C}, \mathbf{D}]$

In other words: for any $G : \widetilde{\mathbf{C}} \longrightarrow \mathbf{D}$ and $\gamma : F \longrightarrow GK$, there exists a unique natural transformation $\alpha : \mathrm{Lan}_K(F) \longrightarrow G$ such that $\alpha_K \circ \eta = \gamma$:

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, Rightarrow, "\eta"'] & \mathbf{D} \\
& \widetilde{\mathbf{C}} \arrow[ur, "\mathrm{Lan}_K(F)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, Rightarrow, "\gamma"'] & \mathbf{D} \\
& \widetilde{\mathbf{C}} \arrow[ur, "G"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbf{C} \arrow[rr, "F"] \arrow[dr, bend right, "K"'] & {} \arrow[d, Rightarrow, bend right, "\eta"'] & \mathbf{D} \\
& \widetilde{\mathbf{C}} \arrow[ur, bend left, "\mathrm{Lan}_K(F)"] \arrow[ur, bend right=40, "G"'] &
\end{tikzcd}$$

$$\alpha$$

NB Kan extensions are unique up to unique isomorphism, which is why we commonly use a definite article (*the* [left Kan extension of F along K]) to refer to them.

Theorem — B.1.11 - Existence of Kan extensions along a functor into a cocomplete category.

Let $\mathbb{C}$ be a **small** category, and $K : \mathbb{C} \longrightarrow \widetilde{\mathbf{C}}$, $F : \mathbb{C} \longrightarrow \mathbf{D}$ be functors.
If $\mathbf{D}$ is **cocomplete**, $\mathrm{Lan}_K(F)$ exists and can be defined, for all $\tilde{C} \in \widetilde{\mathbf{C}}$, as:

$$\underset{K}{\mathrm{Lan}}(F)(\tilde{C}) := \mathrm{colim}\left(K \downarrow \tilde{C} \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathbf{D}\right)$$

On top of that, if F is **fully faithful**, the natural transformation $\eta : F \longrightarrow \mathrm{Lan}_K(F) \circ K$ is an isomorphism.

With the machinery of Kan extensions, presheaves being colimits of representables and Theorem 4.1 are straightforward corollaries of Theorem B.1.11. On top of that, we can express Kan extensions as coends (Theorem B.1.12), which in turn implies the co-Yoneda lemma (Corollary B.1.13).

4.2 Nerve construction: $\mathcal{E} \to \widehat{\mathbb{P}_+}$

In a 2008 article on the n-Category Café titled 'How I Learned to Love the Nerve Construction', Tom Leinster said:

> The nerve construction is inherent in the theory of categories.

And quite understandably: the nerve construction is an application of the Kan extension apparatus which unifies various parts of fields such as (higher) category theory, (higher) homotopy theory, algebraic topology, algebraic geometry, . . . among others. To quote what Urs Schreiber wrote on the the corresponding nLab entry:

> Pretty much every notion of category and higher category comes, or should come, with its canonical notion of simplicial nerve [. . .]

And in our case, the nerve construction is precisely what will enable us to see event structures as presheaves over finite partial orders of events.

4.2.1 Nerve-Realisation paradigm

The general setting is the following:

Definition 4.2.1 — Nerve-Realisation paradigm
Let $F : \mathbb{C} \longrightarrow \mathbf{D}$ be a functor from a small category to locally small cocomplete one.

- The left Kan extension of F along $\mathbf{y}_{\mathbb{C}}$ (the existence of which is due to Theorem B.1.11) is referred to as **Yoneda extension** or **the realisation functor** of F
- It has a right adjoint $\mathrm{N}_F := \underbrace{D \mapsto \mathrm{Hom}_{\mathbf{D}}\left(F(-), D\right)}_{\text{denoted by } \mathrm{Hom}_{\mathbf{D}}(F(=),-)}$ called the **nerve of** F:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] & \mathbf{D} \arrow[dl, bend left=40, "\mathrm{N}_F"] \\
\widehat{\mathbb{C}} \arrow[ur, bend left=20, "\mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

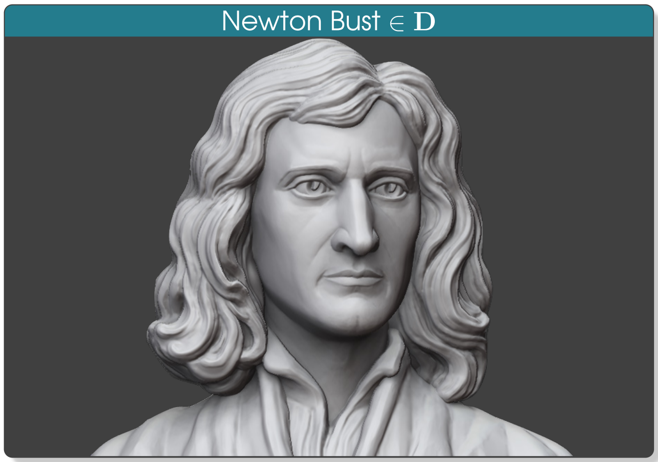
Proposition — B.2.1. With the above notations:

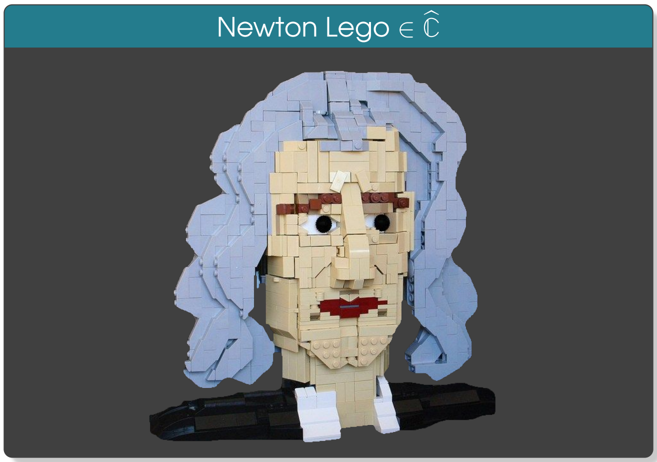
$$\underset{\mathbf{y}_{\mathbb{C}}}{\mathrm{Lan}}(F) \dashv \mathrm{N}_F \cong \underset{F}{\mathrm{Lan}}(\mathbf{y}_{\mathbb{C}})$$

To grasp an intuition for the nerve-realisation paradigm, let's bring back our Lego blocks example. You can think of $\mathbb{C}$-objects as being Lego blocks – and thus $\widehat{\mathbb{C}}$-objects as being Lego constructions due to $\widehat{\mathbb{C}}$ being the free cocompletion of $\mathbb{C}$ – and $\mathbb{D}$-objects as being real-world physical objects. Then:

- the functor F turns each Lego block into a real-world object (possibly a piece of a bigger object that would be colimit in $\mathbf{D}$).
- the realisation functor of F takes a Lego construction and replaces each of its Lego block by their real-world counterpart given by F
- the nerve functor of F associates to every real-world object the "closest matching" Lego construction for this object, by giving a way to probe the object with every Lego block. Indeed, $\mathrm{N}_F(d) := c \mapsto \mathrm{Hom}_{\mathbf{D}}\left(F(c), d\right)$ stores all the information with regard to "embedding" each Lego block c into the real-world object d

For example, our child may happen to have a bust of Newton in his bedroom (the bust being an object of $\mathbf{D}$ in our comparison), and may suddenly feel like making a Lego copy of it, thereby acting like the nerve:


Newton Bust $\in \mathbf{D}$


Newton Lego $\in \widehat{\mathbb{C}}$

Beyond the Lego analogy, $\mathbb{C}$ can really be seen as a category of basic ”shapes” on the basis of which the realisation functor builds and the nerve ”approximates” objects of $\mathbf{D}$. So we would like the nerve to be fully faithful, so that $\mathbf{D}$ is equivalent to a full subcategory of $\widehat{\mathbb{C}}$ (namely: the subcategory of presheaves isomorphic to an object of $\mathrm{N}_F[\mathbf{D}] \subseteq \widehat{\mathbb{C}}$), which would then mean that our Lego construction mechanism is sufficiently accurate to faithfully capture all the ways to transform a given real-world object into another one. Another way to see it is that if the nerve is fully faithful, $\mathrm{N}_F(d) \cong \mathrm{N}_F(d') \iff d \cong d'$ for all $d \in \mathbf{D}$, so we have injectivity on objects ”up to isomorphism”, that is: the category $\mathbf{D}$ is embeds into $\widehat{\mathbb{C}}$ ”up to isomorphism”.

What about event structures? As mentioned before, we do have a nerve construction in this setting as well.

4.2.2 Nerve of the inclusion of finite paths into event structures

Recall that the category of paths $\mathbb{P}$ is the category of finite posets seen as event structures (called *elementary event structures*, *path shapes* or simply *paths*) and rigid maps. As a matter of fact:

- a rigid map can be thought of as extending a path to another one.
- a presheaf A over $\mathbb{P}$ corresponds to a gluing of paths (as $\widehat{\mathbb{P}}$ is the free cocompletion of $\mathbb{P}$). As A is a colimit of representables and for all $P, Q \in \mathbb{P}$, $\mathbf{y}_{\mathbb{P}}(P)(Q) = \mathrm{Hom}_{\mathbb{P}}\,(Q, P)$ describes all the ways to embed Q into P, it follows that for all $P \in \mathbb{P}$, $A(P)$ can be thought of as the set of the states associated to the path P, that is: all the P-shaped computation paths that can be run by the process embodied by A.

But *all* the presheaves over $P \in \mathbb{P}$ are not relevant: for a given $A \in \widehat{\mathbb{P}}$, there should be only one computation path of shape $\emptyset$. So we ought to enforce $A(\emptyset) = \mathbb{1}$. That is what leads us to consider the category of presheaves $A \in \widehat{\mathbb{P}}$ such that $A(\emptyset) = \mathbb{1}$ (such presheaves are said to be *rooted*), denoted by $\widehat{\mathbb{P}}'$. And this category is equivalent to $\widehat{\mathbb{P}_+}$, where $\mathbb{P}_+$ is the category of non-empty paths (Proposition B.2.2).

As in [SW10], the nerve of the inclusion functor $\mathbb{P}_+ \overset{I_+}{\hookrightarrow} \mathcal{E}$ enables us to regard event structures as presheaves over non-empty paths: but is N_{I_+} fully faithful? To answer this, the discussion in appendix brings us to consider the *density* of the inclusion functor I_+.

4.3 Density of non-empty paths $\mathbb{P}_+$ in $\mathcal{E}$

Definition 4.3.1 A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is said to be **dense/codense** if every C-object is a **canonical colimit/limit** of objects of $F[\mathbf{C}]$, *i.e.* for all $C \in \mathbf{C}$,

$$C \;\cong\; \operatorname{colim}\left(F \downarrow C \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D}\right) \quad / \quad C \;\cong\; \lim\left(C \downarrow F \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D}\right)$$

And we can show, in our case, that the inclusion of $\mathbb{P}_+$ in $\mathcal{E}$ is indeed dense:

Theorem — B.3.1. The inclusion functor $\mathbb{P}_+ \overset{I_+}{\hookrightarrow} \mathcal{E}$ is dense.

4.3.1 Sufficient condition for full- and faithfulness of the nerve

Finally, the fact that the density of a functor implies that its nerve is dense can be obtained as a corollary of the following theorem:

Theorem — B.3.2. If $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, $G : \mathbf{C} \longrightarrow \mathbf{D}$ a **continuous** functor and $\mathbf{A} \overset{i}{\hookrightarrow} \mathbf{C}$ a co-dense subcategory, there is a unique extension of every natural transformation $\alpha i : Fi \longrightarrow Gi$ to a natural transformation $\alpha : F \longrightarrow G$.

Corollary — B.3.3 If $\mathbb{C} \overset{i}{\hookrightarrow} \mathbf{D}$ is dense, the nerve functor N_i is fully faithful.

To reuse the Lego analogy, $\mathbb{C}$ being dense in $\mathbf{D}$ can be understood as the the Lego bricks being so small (let's say of atomic size!) that the Lego constructions are faithful enough to distinguish any two non-isomorphic real-world objects!

In the end, we straightforwardly deduce, due to the above corollary:

Lemma — B.3.4. The nerve functor for the embedding $I_+ : \mathbb{P}_+ \hookrightarrow \mathcal{E}$ is full and faithful.

4.4 Conclusion

On the whole, we first saw that how event structures, an operational model of concurrency, are related to special kinds of Scott domains: finitary prime algebraic ones. The configurations of an event structures form a finitary prime algebraic domain, and reciprocally, the set of complete primes of a finitary prime algebraic domain can be given the structure of an event structure. Prime algebraic domains are of crucial importance when it comes to dealing with denotational semantics, which would be the next step.

Second, the concept of free cocompletion was presented, and we sketched some reasons as to why it stems from presheaves, before hinting at a proof using the machinery of Kan extensions.

Lastly, we saw how Kan extensions are involved in the nerve realisation paradigm, and exhibited event structures as presheaves over non-empty paths via the nerve construction.

This was nothing but a tiny step toward studying event structures as presheaves: a lot remains to be done, the long-term objective being to investigate them through the lens of the theory of algebraic effects.

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Appendix

A. Category of event structures

A.1 Event structures

Notation A.1. *We denote the set of all consistent subsets of an event structure E by $\overline{\mathrm{Con}}_E$.*

Lemma A.1.1 Consistent (possibly infinite) sets satisfy the augmentation property and are closed under the subset relation.

Proof

1. Closure under subsets: for every $X \in \overline{\mathrm{Con}}_E$ and $Y \subseteq X$ we have $Y \in \overline{\mathrm{Con}}_E$.

   Consider any consistent set $X \in \overline{\mathrm{Con}}_E$, and consider any $Y \subseteq X$. For every finite subset $Z \subseteq Y$, we have $Z \subseteq X$ and as X is consistent, $Z \in \mathrm{Con}_E$ is consistent. Therefore Y is consistent.

2. The augmentation property holds for $\overline{\mathrm{Con}}_E$: if $e' \leq e \in X \in \overline{\mathrm{Con}}_E$, then $X \cup \{e'\} \in \overline{\mathrm{Con}}_E$.

   Indeed, for all $Y \in \mathcal{P}_{\mathrm{fin}}(X \cup \{e'\}$, either $Y \subseteq_{\mathrm{fin}} X$ or $Y = Y' \cup \{e'\}$ with $Y' \subseteq_{\mathrm{fin}} X$:

   (a) when $Y \subseteq_{\mathrm{fin}} X$: then $Y \in \mathrm{Con}_E$ as X is consistent

   (b) when $Y := Y' \cup \{e'\}$, where $Y' \subseteq_{\mathrm{fin}} X$. Since $e \in X$, we have $Y' \cup \{e\} \subseteq_{\mathrm{fin}} X$, and it follows that $Y' \cup \{e\} \in \mathrm{Con}_E$ as X is consistent. Therefore, $Y' \cup \{e\} \cup \{e'\} \in \mathrm{Con}_E$ by the augmentation property, and finally $Y' \cup \{e\} \cup \{e'\} \supseteq Y' \cup \{e'\} = Y \in \mathrm{Con}_E$ by closure under subsets. ■

Lemma A.1.2 Let E be an event structure. For all $p \in \underline{E}$, the downward closure is a configuration: $\forall p \in \underline{E}, \quad \downarrow p \in \mathcal{C}^o E$. For every consistent $X \in \mathrm{Con}_E$, its downward closure $\downarrow X$ is a configuration. In that case, $\downarrow X$ is finite.

Proof

It suffices to prove the suffix of the theorem, as the prefix follows by choosing $X := \{p\} \in \mathrm{Con}_E$. Consider any consistent $X \in \mathrm{Con}_E$.

This stems from the fact that $\downarrow X$ is:

- finite: Because X is consistent, it is finite, and because $\leq_E$ is well-founded, each $\downarrow p$ is finite. Therefore $\downarrow X = \bigcup_{p \in X} \downarrow p$ is a finite union of finite subsets, hence finite.
- down-closed: by transitivity of $\leq_E$.
- For consistency, let $k := |X| \in \mathbb{N}$ be the cardinality of X, and $n := |(\downarrow X) \setminus X|$ the cardinality of the rest of $\downarrow X$. Consider any enumeration $\# : k + n = |\downarrow X| \xrightarrow{\cong} X$ such that its restriction to the first k elements is an enumeration $\# \big|_k : k \xrightarrow{\cong} X$. Denote $X_i := \{\#j | 0 \leq j \leq i\}$, for every $k \leq i \leq n + k$. We'll show by induction on $i \leq k + n$ that X_i is consistent.
  - $X_k = X \in \mathrm{Con}_E$.
  - Consider any $i + 1 \leq n$, $k \leq i$, and assume $X_i \in \mathrm{Con}_E$. As $\#(i + 1) \in \downarrow X$, there is some $p \in X$ such that $\#(i + 1) \leq p$. By fiat, there is some $0 \leq j \leq k$ such that $p = \#j \in X_j \subseteq X_i$. We use the augmentation property to deduce that $X_{i+1} = X_i \cup \{\#(i + 1)\} \in \mathrm{Con}_E$.

  By induction, $\downarrow X = X_{k+n} \in \mathrm{Con}_E$. ■

Lemma A.1.3 A (finite) subset of events is consistent iff its downward closure is a (finite) configuration.

Proof

Consider any subset $X \subseteq \underline{E}$.

For $\implies$, take any finite $Y \subseteq {\downarrow} X$. For every $b \in Y$ there is some $a \in X$ such that $b \leq a$, let $f : Y \to X$ be a choice function witnessing this fact, and set $Z := f[Y] \subseteq X$, and note that $Y \subseteq {\downarrow} Z$, and that Z is finite and consistent (every subset of a consistent set remains consistent), thus $Z \in \mathrm{Con}_E$. Therefore, by the previous lemma ${\downarrow} Z$ is a configuration, hence consistent, and so $Y \subseteq Z$ is consistent. We've shown that every finite $Y \subseteq {\downarrow} X$ is consistent, and so ${\downarrow} X$ is consistent.

By definition, ${\downarrow} X$ is down-closed, and so ${\downarrow} X$ is a configuration.

For $\impliedby$, note that $X \subseteq {\downarrow} X$ and so if ${\downarrow} X$ is a configuration (hence consistent), it is consistent.

Finally, if one of X and ${\downarrow} X$ is finite, then so is the other by the previous lemma/downward closure of finite sets. ■

Lemma A.1.4 A (finite) subset of events is consistent iff it occurs as a subset of a (finite) configuration.

Proof

Consider any $X \subseteq \underline{E}$. By the previous lemma, X is consistent iff ${\downarrow} X$ is a configuration.

For $\implies$, we are done as $X \subseteq {\downarrow} X$. For $\impliedby$, X is consistent since it is a subset of a consistent set.

Similarly when X is finite. ■

Lemma A.1.5 For all $e, e' \in \underline{E}$, we have: $e \leq e' \implies (\forall x \in \mathcal{C}^o E, e' \in x \implies e \in x)$

Proof

- $\implies$: it is due to the fact that x is down-closed, as a configuration.
- $\impliedby$: with $x := {\downarrow} e'$, it follows that $e \in {\downarrow} e'$, *i.e.* $e \leq e'$ ■

Note that configurations are not closed under unions:

Example A.1 Let $\underline{E} := \{a, b\}$, consider E as a discrete poset, *i.e.* $a \not\leq b$ and $b \not\leq a$, and set $\mathrm{Con}_E := \{\{a\}, \{b\}, \emptyset\}$. Then $\{a\}, \{b\} \in \mathcal{C}^o E$, but $\{a\} \cup \{b\} \notin \mathcal{C}^o E$ because $\{a\} \cup \{b\} \notin \mathrm{Con}_E$. ■

A.2 Maps of event structures

A.2.1 Basic properties

Fix a map of event structures $f : E \to F$.

Lemma A.2.1 $\forall e \in \underline{E}, \quad {\downarrow} fe \subseteq f[{\downarrow} e]$

Proof

${\downarrow} fe$ is the smallest configuration containing fe, since:

- it is a configuration; and
- for every configuration y such that $fe \in y$, ${\downarrow} fe \subseteq y$ as y is down-closed;

Moreover:

- as f preserves configurations, $f[\downarrow e]$ is also a configuration; and
- $fe \in f[\downarrow e]$

so the result follows by minimality of $\downarrow fe$. ∎

Lemma A.2.2 $\forall x \subseteq \underline{E}, \quad \downarrow f[x] \subseteq f[\downarrow x]$

Proof

$$\downarrow f[x] = \downarrow f\Big[\bigcup_{e\in x} e\Big] \overset{\text{direct images preserve union}}{\overset{\downarrow}{=}} \downarrow \bigcup_{e\in x} f(e) \underset{\text{definition of } \downarrow \text{ on sets}}{\underset{\uparrow}{=}} \bigcup_{e\in x} \downarrow f(e) \overset{\text{the previous lemma}}{\overset{\downarrow}{\subseteq}} \bigcup_{e\in x} f[\downarrow e] = f\Big[\bigcup_{e\in x} \downarrow e\Big] \overset{\text{by definition of } \downarrow x}{\overset{\downarrow}{=}} f[\downarrow x]$$ ∎

Lemma A.2.3 Every map $f : E \to F$ is *causality reflecting* on configurations: $\forall e_1, e_2 \in x \in \mathcal{C}^o E, \quad fe_1 \leq_F fe_2 \Longrightarrow e_1 \leq_E e_2$

Proof
If $f(e_1) \in \downarrow f(e_2) \subseteq f(\downarrow e_2)$, then there exists $e_0 \leq_E e_2$ such that $f(e_1) = f(e_0)$. As the configuration x is down-closed: $e_0 \in x$. On top of that, as f is injective on the configuration $x \ni e_1, e_0$, it follows that $e_1 = e_0$; and finally $e_1 = e_0 \leq_E e_2$ ∎

NB A map is not necessarily causality reflecting on the whole carrier set, as exemplified by the following counter-example: if

- $\underline{E} := \{a, b\} \quad \mathrm{Con}_E := \{\{a\}, \{b\}, \emptyset\} \quad a \not\leq_E b$
- $\underline{F} := \{c\} \quad \mathrm{Con}_F := \{\{c\}, \emptyset\}$

then the only map $f : E \to F$ (sending all the elements to c) is not causality reflecting: $c = f(a) \leq_F f(b) = c$ but $a \not\leq b$.

Lemma A.2.4 Note that a function $f : \underline{E} \to \underline{F}$ is

- monotone *iff* $\forall e \in \underline{E},\ f[\downarrow e] \subseteq \downarrow f(e)$ *iff* $\forall x \subseteq \underline{E},\ f[\downarrow x] \subseteq \downarrow f[x]$
- causality reflecting on configurations *iff* $\forall x \in \mathcal{C}^o E, \forall e \in x,\ f_{|x}^{-1}[\downarrow f(e)] \subseteq \downarrow e$

Proof
The proof is straightforward. The equivalence

$$\forall e \in \underline{E},\ f[\downarrow e] \subseteq \downarrow f(e) \iff \forall x \subseteq \underline{E},\ f[\downarrow x] \subseteq \downarrow f[x]$$

is shown analogously to Lemma A.2.2. ∎

Lemma A.2.5 Every rigid map $f : E \to F$ is configuration reflecting on configurations:

$$\forall x' \subseteq x \in \mathcal{C}^o E,\ f[x'] \in \mathcal{C}^o F \Longrightarrow x' \in \mathcal{C}^o E$$

Proof
Let us show that every $x' \subseteq x \in \mathcal{C}^o E$ such that $f[x'] \in \mathcal{C}^o F$ is a configuration of E.

- x' is consistent, as x is (and $\subseteq$ is transitive)

- x' is down-closed: as $f[x']$ is a configuration, hence down-closed:

  $$\downarrow f[x'] \subseteq f[x']$$

  But f being monotone implies that $f[\downarrow x'] \subseteq \downarrow f[x']$.
  So

  $$f[\downarrow x'] \subseteq f[x']$$

  and as taking preimages preserves inclusion and f is injective on the configuration $x \supseteq x'$:

  $$\downarrow x' \overset{\substack{\text{injectivity}\\ \downarrow}}{=} f^{-1}(f[\downarrow x']) \subseteq f^{-1}(f[x']) \overset{\substack{\text{injectivity}\\ \downarrow}}{=} x'$$

■

Lemma A.2.6 A map $f : E \to F$ is rigid iff

$$\forall x \in \mathcal{C}^{o}E, \forall y \in \mathcal{C}^{o}F, \quad y \subseteq f[x] \Longrightarrow \exists x' \in \mathcal{C}^{o}E,\ x' \subseteq x \wedge fx' = y$$

Proof

- $\Longrightarrow$: let us assume that f is rigid and $y \subseteq f[x]$. Then $y := f[x']$, where $x' \in \mathcal{P}_{\text{fin}}(x)$. All that remains to be shown is that x is a configuration, which directly results from the previous lemma.
- $\Longleftarrow$: it'll be sufficient to show that for all $e \in \underline{E}$, $f[\downarrow e] \subseteq \downarrow f(e)$. As f is a map:

  $$y := \downarrow f(e) \subseteq f[\downarrow e]$$

  So there exists $\mathcal{C}^{o}E \ni x' \subseteq \downarrow e$ such that $fx' = y$. On top of that, $f(e) \in y$ so by injectivity of f on the configuration $\downarrow e$: $e \in x'$. Therefore:

  $$\downarrow e \overset{\substack{e \in x' \\ x' \text{ is down-closed} \\ \downarrow}}{\subseteq} x' \underset{\substack{\uparrow \\ \text{images preserve inclusion}}}{\Longrightarrow} f[\downarrow e] \subseteq \underbrace{f[x']}_{= y := \downarrow f(e)}$$

■

Lemma A.2.7 A function $f : \underline{E} \to \underline{F}$ is a rigid map iff:

- it is locally injective;
- it preserves consistency: $X \in \mathrm{Con}_E \implies f[X] \in \mathrm{Con}_F$; and
- for every finite configuration $x \in \mathcal{C}^{o}E$ we have $f[x] = \downarrow f[x]$.

Proof
For $\Longrightarrow$, use Lemma A.2.4 and Lemma A.2.2. To show that f preserves consistency, take any consistent $X \in \mathrm{Con}_E$. Then $\downarrow X$ is a configuration (by Lemma A.1.2), hence $f[\downarrow X]$ is a configuration, and in particular, consistent. Therefore:

$$f[X] \overset{\substack{X \subseteq \downarrow X \\ \downarrow}}{\subseteq} f[\downarrow X] \in \mathrm{Con}_F$$

and $f[X] \in \mathrm{Con}_F$ (by Lemma A.1.1), thus f is consistency preserving.

Assume $\forall x \in \mathcal{C}^o E. f[x] = \downarrow f[x]$. Consider any finite configuration $x \in \mathcal{C}^o E$ in E. By our assumption, $f[x] = \downarrow f[x]$ (so $f[x]$ is down-closed) and $f[x]$ is consistent. Thus it is a configuration. Therefore f is configuration preserving. As it is locally injective, it is therefore a map of event structures. By Lemma A.2.4, as $\downarrow e \in \mathcal{C}^o E$ for all $e \in \underline{E}$, it is monotone. ■

The following example shows we cannot drop the consistency preservation condition:

Example A.2 Consider the following data:

- Let $\underline{E} := \{b < a\}$ and $\mathrm{Con}_E := \mathcal{P}(\underline{E})$.
- Let $\underline{F} := \{a' \neq b'\}$ with the discrete order, and $\mathrm{Con}_F := \{\{a'\}, \{b'\}, \emptyset\}$, i.e., $a' \sim\sim b'$.
- Define $f : \underline{E} \to \underline{F}$ by:

$$f := \begin{cases} a & \mapsto a' \\ b & \mapsto b' \end{cases}$$

and check that $\forall x \in \mathcal{C}^o E := \{\{a, b\}, \{a\}, \{b\}, \emptyset\},\ f[x] = \downarrow f[x]$

then $\{a, b\} \in \mathcal{C}^o E$, but $f[\{a, b\}] = \{a', b'\} \notin \mathcal{C}^o F$, and so f is not configuration preserving. As f is injective, the counter-example holds even when we have (local) injectivity. ■

A.2.2 Well-founded posets

Lemma A.2.8 Let P be a well-founded poset. Then $\langle \underline{P}, \leq_P, \mathcal{P}_{\mathrm{fin}}(P) \rangle$ is an event structure.

Proof
All the properties to check are straightforwardly true:

- $\leq_P$ is indeed a partial order for which all the downward closures are finite (as P is well-founded)
- all the singletons are members of $\mathcal{P}_{\mathrm{fin}}(P)$
- $\mathcal{P}_{\mathrm{fin}}(P)$ is stable under inclusion
- the *augmentation property* holds, as $x \cup \{e'\}$ is finite provided x is.

■

Lemma A.2.9 Let P, Q be well-founded posets, and $f : \underline{P} \to \underline{Q}$ be any function. Then: f is a rigid map iff for all $p \in \underline{P},\ f[\downarrow p] = \downarrow fp$ and f is causality reflecting on configurations.

Proof

- $\Longrightarrow$: if f is a rigid map, then it is a map, and it is causality reflecting on configurations as showed previously, and for all $p \in \underline{P}$:
  - $f[\downarrow p] \subseteq \downarrow fp$ as f is monotone
  - $\downarrow fp \subseteq f[\downarrow p]$ as shown previously (since $\downarrow fp$ is the smallest configuration containing fp)
- $\Longleftarrow$: if f is causality reflecting on configurations and $f[\downarrow p] = \downarrow fp$, then:
  - f is monotone as $f[\downarrow p] \subseteq \downarrow fp$ for all $p \in \underline{P}$
  - f is configuration preserving, as the image of a finite set is a finite set

- f is locally injective: let $p, p' \in x \in \mathcal{C}^o P$ such that $f(p) = f(p')$. Since $x \in \mathcal{C}^o P$ and $f(p) \in {\downarrow} f(p) \in \mathcal{C}^o Q$ (by Lemma A.1.2 and as Q is well-founded), then as f is causality reflecting on configurations:
  - $f(p) \leq_F f(p')$ implies that $p \leq_E p'$
  - $f(p') \leq_F f(p)$ implies that $p' \leq_E p$

  thus $p = p'$ by antisymmetry. ■

A.2.3 Path-lifting property

Reminder

Definition A.2.1 — The comma category $F \downarrow G$ of $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$ is the category whose

- **objects** are triples $\langle A, B, \phi \rangle$ where $A \in \mathbf{A}$, $B \in \mathbf{B}$ and $\phi : FA \to GB$
- **morphisms** from $\langle A, B, \phi \rangle$ to $\langle A', B', \phi' \rangle$ are pairs $\langle f, g \rangle$ where $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ such that the evident square commutes:

$$\begin{tikzcd}
FA \arrow[r, "\phi"] \arrow[d, "Ff"'] & GB \arrow[d, "Gg"] \\
FA' \arrow[r, "\phi'"'] & GB'
\end{tikzcd}$$

NB A common special case is when G (resp. F) is a constant functor Δ_C, in which case the comma category is called the *slice* (resp. *coslice*) category over C, denoted by $F \downarrow C$ (resp. $C \downarrow G$) by abuse of notation.

In the *slice* category $F \downarrow C$:

- **objects** are pairs $\langle FA, \phi \rangle$ where $A \in \mathbf{A}$ and $\phi : FA \to C$
- **morphisms** are of the form $f : \langle FA, \phi \rangle \longrightarrow \langle FA', \phi' \rangle$ where $f : A \to A'$ and the following triangle commutes:

$$\begin{tikzcd}
FA \arrow[rr, "f"] \arrow[dr, "\phi"'] & & FA' \arrow[dl, "\phi'"] \\
& C &
\end{tikzcd}$$

Definition A.2.2 — The category of paths $\mathbb{P}$ is the category whose

- **objects** are finite posets (seen as event structures)
- **morphisms** are rigid maps

$I : \mathbb{P} \hookrightarrow \mathcal{E}$ is its full embedding in the category of event structures and rigid maps.

Definition A.2.3 — A map $f : E \to F$ satisfies the *path lifting property* iff every such commuting square where $P, Q \in \mathbb{P}$ factors into two commuting triangles:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & E \arrow[d, "f"] \\
Q \arrow[r] \arrow[ur, dotted] & F
\end{tikzcd}$$

NB The path lifting property asserts the existence but not necessarily uniqueness of the diagonal map.

Lemma A.2.10 We can regard $\langle \mathcal{C}^o E, \subseteq \rangle$ as a sub-category of the comma $I \downarrow E$, which is **not full**.

Proof

Let's show that the subcategory is not full: with $\underline{E} := \{\alpha, \beta\},\ \alpha \not\leq \beta$ $\qquad x := \{\alpha\} \in \mathcal{C}^o E$ $\qquad f :=$

$$\begin{cases} x & \to E \\ \alpha & \mapsto \beta \end{cases}$$

then the following diagram commutes:

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "f"'] & & E \arrow[dl, "id"] \\
& E &
\end{tikzcd}$$

so that $f \in \mathrm{Hom}_{I \downarrow E}\,(x, E)$, but f is not the inclusion map. ■

Lemma A.2.11 A rigid map $f : E \to F$ satisfies the path lifting property iff:

$$\forall x \in \mathcal{C}^o E, \forall y \in \mathcal{C}^o F, \quad f[x] \subseteq y \Longrightarrow \exists x' \in \mathcal{C}^o E;\ x \subseteq x' \ \wedge\ f[x'] = y$$

Proof

- $\Longrightarrow$: Let $x \in \mathcal{C}^o E$ and $y \in \mathcal{C}^o F$ such that $f[x] \subseteq y$. Then by the path lifting property, the following square factors into these two triangles (where ι_x and ι_y are the inclusion maps):

$$\begin{tikzcd}
x \arrow[r, "\iota_x"] \arrow[d, "f_{|x}"'] & E \arrow[d, "f"] \\
y \arrow[r, "\iota_y"'] \arrow[ur, dotted, "\exists\, \phi"] & F
\end{tikzcd}$$

  By setting $x' := \phi[y]$:

  - $x' \in \mathcal{C}^o E$: since it is the image of a finite configuration by a configuration-preserving map
  - $x \subseteq x'$: as

$$\begin{aligned} & f[x] \subseteq y \\ \Longleftrightarrow \quad & \underbrace{\phi[f[x]]}_{= \iota_x[x] = x} \subseteq \underbrace{\phi[y]}_{:= x'} \qquad \text{(since taking images preserves inclusion)} \end{aligned}$$

  - $f[x'] = f[\phi[y]] = \iota_y(y) = y$

- $\Longleftarrow$: let $P, Q \in \mathbb{P}$ such that the following square commutes:

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "g"'] & E \arrow[d, "f"] \\
Q \arrow[r, "q"'] & F
\end{tikzcd}$$

  Then $x := p[P] \in \mathcal{C}^{\text{o}}E$ and $y := q[Q] \in \mathcal{C}^{\text{o}}F$ (images of finite configurations by configuration-preserving maps) are such that

$$f[x] = f[p(P)] = q[g[P]] \overset{g[P] \subseteq Q}{\subseteq} q[Q] := y$$

  so there exists a finite configuration $\mathcal{C}^{\text{o}}E \ni x' \supseteq x$ such that $f[x'] = y$.

  By local injectivity, the function $f_{|x'}^{|y} : x' \to y$ (corestriction to y of the restriction to x') is **bijective**. One can check that its inverse ${f_{|x'}^{|y}}^{-1}$ remains a rigid map:

  - it is injective.
  - it is monotone, since the map f is causality reflecting on configurations (as seen in Lemma A.2.3).
  - it is configuration-preserving, that is:

$$\forall \mathcal{C}^{\text{o}}F \ni y' \subseteq y,\ f^{-1}[y'] \in \mathcal{C}^{\text{o}}E$$

    which amounts to show that

$$\forall x'' \subseteq x',\ y' := f[x''] \in \mathcal{C}^{\text{o}}F \Longrightarrow x'' \in \mathcal{C}^{\text{o}}E$$

    which results from f being configuration-reflecting on the configuration x' (as it is a rigid map: cf.Lemma A.2.5).

  So by setting $\phi := {f_{|x'}^{|y}}^{-1} \circ q$, the result follows, as

$$\begin{aligned}
\phi \circ g &= {f_{|x'}^{|y}}^{-1} \circ \underbrace{q \circ g}_{= f \circ p} \\
&= p & \text{(as } f \circ p : P \to f[x] \subseteq f[x'] = y\text{)}
\end{aligned}$$

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "g"'] & E \arrow[d, "f"] \\
Q \arrow[r, "q"'] \arrow[ur, dotted, "\phi"'] & F
\end{tikzcd}$$

■

A.3 Prime algebraicity

This section is based on Sec. 2 of Winskel's paper [Win09].

A.3.1 Consistently complete posets

Proposition A.3.1 The empty poset is a consistently complete poset.

Proof
It suffices to show that the there is no finitely bounded subset in $\underline{D} := \emptyset$. Indeed, $X := \emptyset$ is the only subset of $\underline{D}$, and the only finite subset thereof is $Y := \emptyset$. But Y has no upper bound in D, as there is no element in $\underline{D} = \emptyset$. ■

Let D be a consistently complete poset.

Lemma A.3.2 If $\underline{D} \neq \emptyset$ then D has a least element: $\bigvee \emptyset$.

Proof
To begin with, let us note that $\emptyset \subseteq \underline{D}$ is finitely bounded. Indeed, for all $y \in \mathcal{P}_{\text{fin}}(\emptyset) = \{\emptyset\}$, any element $d \in D \neq \emptyset$ is an upper bound for $y = \emptyset$ ($\forall d' \in y, d' \leq\ d$ holds since $y = \emptyset$).

Therefore, as D is a consistently complete: the finitely bounded subset $\emptyset \subseteq \underline{D}$ has an upper bound $\bot := \bigvee \emptyset$ in D.

Consequently, $\bot$ has to be a least element, since if were not the case, there would exist another element $d \in \underline{D}$ such that $d < \bot$, which would contradict the minimality of $\bot$ as d is also an upper bound of $\emptyset$ in D. ■

Lemma A.3.3 Every non-empty subset $\emptyset \neq X \subseteq \underline{D}$ has a greatest lower bound.

Proof
First, note that $Z := \bigcap_{x \in X} {\downarrow} x \subseteq \underline{D}$ is finitely bounded: for all $Y \in \mathcal{P}_{\text{fin}}(Z)$, every $x \in X \neq \emptyset$ is an upper bound of Y (by definition of Z).

So

$$\bigwedge X := \bigvee \underbrace{\bigcap_{x \in X} {\downarrow} x}_{= Z}$$

exists, since D is consistently complete. Let us show that it is the greatest lower bound of X. Indeed:

- $\bigwedge X$ **is a lower bound of** X**:** every $x' \in X \neq \emptyset$ is an upper bound of Z, so as $\bigwedge X := \bigvee Z$ is the least upper bound thereof, by definition:

$$\forall x' \in X,\ \bigwedge X \leq x'$$

- $\bigwedge X$ **is the greatest lower bound:** if $x' \in \underline{D}$ is a lower bound of X, then $x' \in \bigcap_{x \in X} {\downarrow} x = Z$ by definition, and as $\bigwedge X := \bigvee Z$ is an upper bound of Z:

$$x' \leq \bigwedge X$$

■

A.3.2 Prime algebraic domains

Notation A.2. *Similarly to the downward closure, we extend the prime downward-closure to sets by setting:*

$$\downarrow^{\mathrm{p}} X := \bigcup_{x \in X} \downarrow^{\mathrm{p}} x$$

Lemma A.3.4 If D is *pointed*, that is, D has a least element $\bot$, then $\bot$ is not a complete prime.

Proof
By setting $X := \emptyset$:

$$\bot \leq \bot \overset{\substack{\text{by Lemma A.3.2} \\ \downarrow}}{=} \bigvee \emptyset = \bigvee X$$

but there exists no element in X, so $\bot$ is not a complete prime. ■

Proposition A.3.5 The unit interval $\mathbb{I} = [0,1]$ is *not* a prime algebraic domain. It has no complete prime.

Proof

- 0 is not a complete prime by Lemma A.3.4, as it is the least element of $\mathbb{I}$.
- no $x \in]0,1]$ is a complete prime either. Indeed, by setting $\emptyset \neq X := [0,x[$,

  $$x \leq \bigvee X = x$$

  but there exists no $x' \in [0,x[$ such that $x \leq x'$.

So the set of complete primes of $\mathbb{I}$ is $\emptyset$, and $\mathbb{I}$ is clearly not a prime algebraic domain as $\mathbb{I} \neq \{0\}$ (so the prime algebraicity property does not hold). ■

Proposition A.3.6 The ordinal $\omega + 1 := \{0 \leq 1 \leq 2 \leq \cdots \leq \omega\}$ is a prime algebraic domain.

Proof

- 0 is not a complete prime by Lemma A.3.4, as it is the least element of $\omega + 1$.
- every $n \in \mathbb{N} \backslash \{0\}$ is a complete prime: if $\emptyset \neq X \subseteq \underline{\omega + 1} := \mathbb{N} \cup \{\omega\}$ (note that all subsets are bounded by ω and X can be assumed to be non empty as $n \neq 0$) and $n \leq \bigvee X$, then there must exist $d \in X$ such that $n \leq d$. Otherwise, by contradiction, we would have

  $$\forall d \in X,\ \mathbb{N} \ni n > d$$

  as $\omega + 1$ is totally ordered, and $X \subseteq \mathbb{N}$ would have to be a finite set, so that

  $$\bigvee X := \sup X \overset{X \text{ is finite}}{=} \max X \in X \neq \emptyset$$

  Thus, $d := \bigvee X \in X$ would be an element such that $n \leq d$ (by hypothesis).
- ω is not a complete prime: by setting $X := \mathbb{N}$ (bounded by ω), $\bigvee X = \omega$ as ω is the only upper bound of X, and

  $$\omega \leq \bigvee X = \omega$$

  but there exists no integer $n \in X := \mathbb{N}$ such that $\omega \leq n$.

So the set of complete primes of $\omega + 1$ is $\mathbb{N} \backslash \{0\}$, and since

- $\omega + 1$ is a *consistently complete poset*: for every subset $X \subseteq \underline{\omega + 1}$ (all subsets are bounded by ω, hence finitely bounded),
  - if $X = \emptyset$, then $\bigvee X = 0 \in \underline{\omega + 1}$
  - if X is finite, then $\bigvee X := \sup X = \max X \in \underline{\omega + 1}$
  - if X is infinite, then the only upper bound of X is ω, so that $\bigvee X := \sup X = \omega \in \underline{\omega + 1}$
- *prime algebraicity property*: for all $d \in \underline{\omega + 1}$:
  - if $d = 0$,
  $$\bigvee {\downarrow^{\mathrm{p}}} d = \bigvee \emptyset = 0$$
  - if $d \in \mathbb{N}\backslash\{0\}$, then
  $$\bigvee {\downarrow^{\mathrm{p}}} d = \bigvee \{p \leq d \mid p \in \mathbb{N}\backslash\{0\}\} = d$$
  - if $d = \omega$, then $\{p \leq \omega \mid p \in \mathbb{N}\} = \mathbb{N}$, and we have indeed $\omega = \bigvee \mathbb{N}$ (as ω is the only upper bound of $\mathbb{N}$).

Therefore, $\omega + 1$ is a prime algebraic domain. ■

Proposition A.3.7 Let $(\omega + 1)^2$ be the poset of pairs from $\omega + 1$ with the componentwise order.

1. We have that $\langle a, b\rangle$ is a complete prime iff $a, b < \omega$, and $a = 0$ or $b = 0$, but not both.
2. $(\omega + 1)^2$ is a prime algebraic domain.

Proof

1. Let $\langle a, b\rangle \in (\omega + 1)^2$:
   (a) *if* $\langle a, b\rangle := \langle 0, 0\rangle$*:* then $\langle a, b\rangle$ is not a complete prime by Lemma A.3.4, as it is the least element of $(\omega + 1)^2$.
   (b) *if* $\langle a, b\rangle \in \mathbb{N}^2$ *and* $a = 0,\ b > 0$*:* then for any $\emptyset \neq X \subseteq \underline{(\omega + 1)^2}$ (all subsets are bounded by $\langle \omega, \omega\rangle$ and X can be assumed to be non empty as $\langle a, b\rangle \neq \langle 0, 0\rangle$) such that $\langle a, b\rangle = \langle 0, b\rangle \leq \bigvee X$, then there must exist $\langle d_1, d_2\rangle \in X$ such that $b \leq d_2$, as the order is pointwise and $b \in \mathbb{N}\backslash\{0\}$ is a complete prime of $\omega + 1$ (cf. previous lemma). So $\langle a, b\rangle = \langle 0, b\rangle \leq \langle d_1, d_2\rangle$ (as trivially $0 \leq d_1 \in \underline{\omega + 1} := \mathbb{N} \cup \{\omega\}$), and $\langle a, b\rangle$ is a complete prime.
   (c) *if* $\langle a, b\rangle \in \mathbb{N}^2$ *and* $a > 0,\ b = 0$*:* symmetrically, we show that $\langle a, b\rangle$ is a complete prime.
   (d) *if* $\langle a, b\rangle \in \mathbb{N}^2\backslash\{\langle 0, 0\rangle\}$*:* by setting $X := \{\langle a, b - 1\rangle, \langle a - 1, b\rangle\}$, $\langle a, b\rangle \leq \bigvee X = \langle a, b\rangle$, but there is no $d \in X$ such that $\langle a, b\rangle \leq d$, so $\langle a, b\rangle$ is not a complete prime
   (e) *if* $a = \omega$*:* by setting $X := \mathbb{N} \times \{b\}$, $\langle a, b\rangle = \langle \omega, b\rangle \leq \bigvee X = \langle \omega, b\rangle$, but there exists no $\langle d_1, d_2\rangle \in X$ such that $\langle \omega, b\rangle \leq \langle d_1, d_2\rangle$ (since there is no integer $d_1 \in \mathbb{N}$ such that $\omega \leq d_1$), so $\langle a, b\rangle$ is not a complete prime.
   (f) *if* $b = \omega$*:* symmetrically, we show that $\langle a, b\rangle$ is not a complete prime.

   Therefore, the set of complete primes of $(\omega + 1)^2$ is
   $$\{\langle a, b\rangle \ \mid\ \langle a, b\rangle \in \mathbb{N}^2\backslash\{\langle 0, 0\rangle\} \wedge \big(a = 0 \vee b = 0\big)\}$$

2. - $(\omega + 1)^2$ is a *consistently complete poset*: for every subset $X \subseteq \underline{(\omega + 1)^2}$ (all subsets are bounded by $\langle \omega, \omega\rangle$, hence finitely bounded),
     - if $X = \emptyset$, then $\bigvee X = \langle 0, 0\rangle \in \underline{(\omega + 1)^2}$

  - else:

$$\bigvee X = \left\langle \sup_{(d_1,d_2)\in X} d_1,\ \sup_{(d_1,d_2)\in X} d_2 \right\rangle$$

    where $\sup_{(d_1,d_2)\in X} d_1,\ \sup_{(d_1,d_2)\in X} d_2 \in \underline{\omega+1}$ (cf. previous lemma), so $\bigvee X \in \underline{(\omega+1)^2}$

- *prime algebraicity property*: for all $d := \langle d_1, d_2\rangle \in \underline{(\omega+1)^2}$:
  - if $d = \langle 0, 0\rangle$,

$$\bigvee {\downarrow^{\mathrm{p}}} d = \bigvee \emptyset = \langle 0, 0\rangle$$

  - if $d \in \mathbb{N}^2 \backslash \{\langle 0, 0\rangle\}$, as

$${\downarrow^{\mathrm{p}}} d = \{\langle a, 0\rangle\}_{0<a\leq d_1} \cup \{\langle 0, b\rangle\}_{0<b\leq d_2}$$

    then $\bigvee {\downarrow^{\mathrm{p}}} d = \langle d_1, d_2\rangle$
  - if $d_1 = \omega$ and $d_2 < \omega$, $\bigvee {\downarrow^{\mathrm{p}}} d = \langle d_1, d_2\rangle$ as well since

$${\downarrow^{\mathrm{p}}} d = \{\langle a, 0\rangle\}_{0<a<\omega} \cup \{\langle 0, b\rangle\}_{0<b\leq d_2}$$

    and $\bigvee \{a \mid 0 < a < \omega\} = \omega$
  - if $d_1 < \omega$ and $d_2 = \omega$ or if $d = \langle \omega, \omega\rangle$, the results follow similarly

On the whole, $(\omega+1)^2$ is indeed a prime algebraic domain.

■

A.3.3 Axiom F

Definition A.3.1 — A directed subset $X \subseteq \underline{D}$ is a non-empty subset such that

$$\forall x, y \in X,\ \exists z \in X;\ x, y \leq z$$

NB Note that directed subsets

- can be equivalently described as sets $X \subseteq \underline{D}$ such that every finite subset of X has an upper bound *in* X
- are finitely bounded, which is why we can talk about their least upper bound.

Definition A.3.2 — An element $i \in \underline{D}$ **is isolated** iff for every *directed* subset $X \subseteq \underline{D}$ we have:

$$i \leq \bigvee X \implies \exists d \in X;\ i \leq d$$

Definition A.3.3 — D **satisfies axiom F** when ${\downarrow} i$ is finite for every isolated element $i \in \underline{D}$.

Definition A.3.4 — D **is finitary** iff ${\downarrow^{\mathrm{p}}} p$ is finite for every complete prime p.

Lemma A.3.8 A prime algebraic domain D satisfies axiom F iff it is finitary.

Proof

- $\Longrightarrow$: every complete prime is an isolated element since directed sets are finitely bounded and finitely bounded sets are bounded in prime algebraic domains, thus it has a finite downward closure by axiom F.
- $\Longleftarrow$: let $i \in \underline{D}$ be an isolated element. By prime algebraicity: $i = \bigvee \downarrow^{\mathrm{p}} i$. Let

  $$I := \left\{ \bigvee P \mid P \in \mathcal{P}_{\text{fin}}(\downarrow^{\mathrm{p}} i) \right\} \supseteq \downarrow^{\mathrm{p}} i$$

  Then by construction, I is directed, and $i = \bigvee I$. So, since i is isolated, there exists a finite subset $P \in \mathcal{P}_{\text{fin}}(\downarrow^{\mathrm{p}} i)$ such that $i \leq \bigvee P$, and as $\bigvee P \leq i$ (i is an upper bound of P):

  $$i = \bigvee \underbrace{P}_{\in \mathcal{P}_{\text{fin}}(\downarrow^{\mathrm{p}} i)}$$

  As a consequence: for all $p' \in \downarrow^{\mathrm{p}} i$, $p' \leq i = \bigvee P$, and as P is bounded (by its lub) and p' is a complete prime, it follows that there exists $p \in P$ such that $p' \leq p$. Thus $\downarrow^{\mathrm{p}} i \subseteq \downarrow^{\mathrm{p}} P := \bigcup_{p \in P} \downarrow^{\mathrm{p}} p$, and $\downarrow^{\mathrm{p}} i = \downarrow^{\mathrm{p}} P$. Therefore $\downarrow^{\mathrm{p}} i = \bigcup_{p \in P} \downarrow^{\mathrm{p}} p$ is finite, as a union of finite (by hypothesis) sets.

  Finally, let us show that $\downarrow i$ is finite. Indeed: for every $x \in \downarrow i$ there exists $P_x \subseteq \downarrow^{\mathrm{p}} i$ such that $x = \bigvee P_x$, by prime algebraicity. And as lowest upper bounds are unique (by antisymmetry of $\leq$), the function

  $$\begin{cases} \downarrow i & \longrightarrow \mathcal{P}(\downarrow^{\mathrm{p}} i) \\ x & \mapsto P_x \end{cases}$$

  is injective, so

  $$|\downarrow i| \leq |\mathcal{P}(\downarrow^{\mathrm{p}} i)| = 2^{\overbrace{|\downarrow^{\mathrm{p}} i|}^{\in \mathbb{N}}}$$

  and $\downarrow i$ is indeed finite.

■

A.4 Event structures and finitary prime algebraic domains

Lemma A.4.1 Let E be an event structure. The poset $\langle \mathcal{C}E, \subseteq \rangle$ is a prime algebraic domain, whose complete primes are the configurations $\{\downarrow e\}_{e \in \underline{E}}$

Proof

First, let us note that the lowest upper bound for sets (partially ordered by the inclusion) is the union (by universal property of the coproduct). If X is a set of sets, we set (by abuse of notation):

$$\bigcup X := \bigcup_{x \in X} x$$

Let us show that the poset $\langle \mathcal{C}E, \subseteq \rangle$ is consistently complete and satisfies the prime algebraicity property.

- $\langle \mathcal{C}E, \subseteq \rangle$ *is consistently complete*: let $X \subseteq \mathcal{C}E$ be a finitely bounded subset. Then by definition, for all $Y \in \mathcal{P}_{\text{fin}}(X)$, there exists a configuration $x_Y \in \mathcal{C}E$ such that: $\forall x \in Y,\ x \subseteq x_Y$.

  Let us show that the lowest upper bound of X in $\mathcal{P}(E)$:

  $$x^* := \bigvee X = \bigcup X = \bigcup_{x \in X} x$$

  is actually a configuration (and the result will follow). Indeed:

  - x^* is down closed as each $x \in X$ is a configuration, hence down-closed.
  - x^* is consistent: if $y \in \mathcal{P}_{\text{fin}}(x^*)$, there exists a finite subset $Y \subseteq_{fin} X$ such that

    $$y \subseteq \bigcup_{x \in Y} x = \bigcup Y = \bigvee Y \overset{\substack{\text{by minimality of the least upper bound} \\ \downarrow}}{\subseteq} x_Y \in \mathcal{C}E$$

    so y is consistent, as a subset of a consistent set.

- *prime algebraicity property*:

  1. Let us show that the set of complete primes is

     $$\{\downarrow e \ \mid \ e \in \underline{E}\}$$

     Indeed:

     - every configuration $\downarrow e$, where $e \in \underline{E}$, is a complete prime: if $X \subseteq \mathcal{C}E$ and $\downarrow e \subseteq \bigvee X = \bigcup_{x \in X} x$, then there exists $x \in X$ such that $e \in x$, and since x is down-closed (as a configuration): $\downarrow e \subseteq x$
     - if a configuration $x \in \mathcal{C}E$ is not of the form $\downarrow e$, for $e \in \underline{E}$, then it is not a complete prime. Indeed, in this case, there exists a subset $x' \subseteq x$, $|x'| > 1$, such that:

       $$x = \bigsqcup_{e \in x'} \downarrow e$$

       so that

       $$x \subseteq \bigcup \{ \underbrace{\downarrow e}_{\in \mathcal{C}E} \}_{e \in x'}$$

       but there exists no $e \in x'$ such that $x \subseteq \downarrow e$ (as $|x'| > 1$).

  2. Every $x \in \mathcal{C}E$ can be written as:

     $$x = \bigcup_{e \in x} \downarrow e = \bigcup \{\downarrow e \ \mid \ e \in \underline{E} \wedge \downarrow e \subseteq x\}$$

     The last equality stems from the fact that:

     $$e \in x \Longleftrightarrow e \in \underline{E} \wedge \downarrow e \subseteq x$$

     ($\Longleftarrow$ is obvious and $\Longrightarrow$ is due to x being down-closed).

■

NB Finite configurations are not enough, as showed by the following counter-example:
Let E be an infinite discrete poset, such that $\mathrm{Con}_E := \mathcal{P}_{\text{fin}}(E)$ (thus $\mathcal{C}^o E = \mathrm{Con}_E$). Then $\langle \mathcal{C}^o E, \subseteq \rangle$ is not consistently complete:

- $X := \{\{e\} \mid e \in \underline{E}\}$ is finitely bounded, since for every $Y \in \mathcal{P}_{\text{fin}}(X)$,

$$y^* := \underbrace{\bigcup_{y \in Y} y}_{\in \mathcal{C}^o E \text{ as } Y \text{ is finite}}$$

  is an upper bound (it's even the least upper bound) of Y.
- however, X has no upper bound in $\mathcal{C}^o E$ (let alone a least upper bound): any upper bound $x^* \subseteq E$ of X would satisfy (by definition):

$$\forall x \in X,\ x \subseteq x^*$$

  so

$$\bigcup_{x \in X} x \subseteq x^*$$

  but $\bigcup_{x \in X} x = \underline{E}$ (by definition of X), so $x^* = E$, and $x^* \notin \mathcal{C}^o E$

Lemma A.4.2 Let D be a prime algebraic domain satisfying axiom F, we have an event structure $\mathfrak{P}_D$ given by:

$$\underline{\mathfrak{P}_D} := \{p \in \underline{D} \mid p \text{ complete prime}\} \qquad p_1 \leq_{\mathfrak{P}_D} p_2 \iff p_1 \leq_D p_2 \qquad x \in \mathrm{Con}_{\mathfrak{P}_D} \iff x \text{ bounded in } \underline{D}$$

Proof
The proof is straightforward:

- $\{p \in \underline{D} \mid p \text{ is a complete prime}\}$ is indeed a partial order for which all the downward closures are finite (by Lemma A.3.8 as D satisfies axiom F)
- singletons are bounded
- subsets of bounded sets remain bounded
- the *augmentation property* holds, since if $p' \leq_{\mathfrak{P}_D} p \in x$ has an upper bound d, $p' \leq_{\mathfrak{P}_D} p \leq_{\mathfrak{P}_D} d$, and d is still an upper bound of $x \cup \{p'\}$.

■

NB To show that a function $f : \underline{C} \to \underline{D}$ is map of prime algebraic domains, one only needs to show that

- f is monotone
- for every finitely bounded $X \subseteq \underline{C}$, $f \bigvee X \leq_D \bigvee f[X]$

as f being monotone implies that for all finitely bounded $X \subseteq \underline{C}$, $f[X]$ is bounded by $f \bigvee X$, thus $\bigvee f[X] \leq_D f \bigvee X$.

Definition A.4.1 — $\mathscr{P}$ denotes the category of prime algebraic domains satisfying axiom F (or equivalently: finitary prime algebraic domains) and their maps.

Notation A.3. *If X is a subset of a poset, we set*

$$\Downarrow X := \{\downarrow x \mid x \in X\}$$

If $\mathscr{X}$ is a set of subsets of a poset, we set

$$\Downarrow\!\!\Downarrow \mathscr{X} := \{\Downarrow X \mid X \in \mathscr{X}\}$$

If X is a subset of a consistently complete poset, we set

$$\Downarrow^{\mathrm{p}} X := \{\downarrow^{\mathrm{p}} x \mid x \in X\}$$

Lemma A.4.3 In $\mathcal{E}$, every event structure $E \in \mathcal{E}$ satisfies:

$$E \simeq \mathfrak{P}_{\mathcal{C}E}$$

Proof
Let us show that

$$\mathfrak{P}_{\mathcal{C}E} = \langle \Downarrow \underline{E}, \subseteq, \Downarrow\!\!\Downarrow \mathrm{Con}_E \rangle$$

Indeed:

- the complete primes of $\mathcal{C}E$ are of the form $\downarrow e$, where $e \in \underline{E}$ (Lemma A.4.1), so

$$\underline{\mathfrak{P}_{\mathcal{C}E}} = \{\downarrow e \mid e \in \underline{E}\} = \Downarrow \underline{E}$$

- the partial order is a restriction of the order of $\mathcal{C}E$, that is the set inclusion $\subseteq$
- for all $X := \Downarrow \underbrace{x}_{:= \{e_1, \dots, e_n\} \subseteq \underline{E}} = \{\downarrow e_1, \dots, \downarrow e_n\} \in \mathcal{P}_{\text{fin}}\left(\underline{\mathfrak{P}_{\mathcal{C}E}}\right)$:

$$\begin{aligned}
& X = \Downarrow x \in \mathrm{Con}_{\mathfrak{P}_{\mathcal{C}E}} & \\
\iff\ & \exists y \in \mathcal{C}E;\ X \text{ is bounded by } y & \text{(by definition of } \mathrm{Con}_{\mathfrak{P}_{\mathcal{C}E}}\text{)} \\
\iff\ & \exists y \in \mathcal{C}E;\ \forall i \in \{1, \dots, n\}\,,\ \downarrow e_i \subseteq y & \\
\iff\ & \exists y \in \mathcal{C}E;\ \forall i \in \{1, \dots, n\}\,,\ e_i \in y & \text{(as the configuration } y \text{ is down-closed)} \\
\iff\ & \exists y \in \mathcal{C}E;\ x = \{e_1, \dots, e_n\} \subseteq y & \\
\iff\ & x \text{ is consistent in } E & \text{(by Lemma A.1.4)} \\
\iff\ & x \in \mathrm{Con}_E & \text{(as } x \text{ is finite)}
\end{aligned}$$

So

$$\mathrm{Con}_{\mathfrak{P}_{\mathcal{C}E}} = \{\Downarrow x \mid x \in \mathrm{Con}_E\} = \Downarrow\!\!\Downarrow \mathrm{Con}_E$$

As a result, $e \mapsto \downarrow e$ is clearly a bijective rigid map (it is a bijection and preserves the configurations) from E to $\mathfrak{P}_{\mathcal{C}E}$. ■

Lemma A.4.4 In $\mathscr{P}$, every prime algebraic domain D satisfies:

$$D \simeq \langle \mathcal{C}\mathfrak{P}_D, \subseteq \rangle$$

Proof
Let us show that

$$\mathcal{C}\mathfrak{P}_D = \Downarrow^{\mathrm{p}} \underline{D}$$

- $\mathcal{C}\mathfrak{P}_D \subseteq \Downarrow^{\mathrm{p}} \underline{D}$: Let $X \in \mathcal{C}\mathfrak{P}_D$, then
  - X is downward-closed in $\mathfrak{P}_D$, that is $X = \downarrow^{\mathrm{p}} X$
  - X is consistent in $\mathfrak{P}_D$: for all $Y \in \mathcal{P}_{\mathrm{fin}}(X)$, Y is bounded in D. Thus X finitely bounded.

  The last point implies that X has an upper bound $\bigvee X$ in $\underline{D}$, as D is consistently complete.

  Let us show that

  $$\overbrace{\downarrow^{\mathrm{p}} X}^{=X} = \downarrow^{\mathrm{p}} \bigvee X$$

  - the inclusion $\subseteq$ stems from $\bigvee X$ being an upper bound of X
  - conversely: if $\downarrow^{\mathrm{p}} \bigvee X \ni p \leq \bigvee X$, then as X is finitely bounded and p is a complete prime, there exists $d \in X$ such that $p \leq d$, so $p \in \downarrow^{\mathrm{p}} X$.

  Therefore, $X = \downarrow^{\mathrm{p}} \bigvee X \in \Downarrow^{\mathrm{p}} \underline{D}$
- $\Downarrow^{\mathrm{p}} \underline{D} \subseteq \mathcal{C}\mathfrak{P}_D$: every $\downarrow^{\mathrm{p}} d \in \Downarrow^{\mathrm{p}} \underline{D}$ is trivially down-closed and is consistent, i.e. finitely bounded in this context, as it is bounded by $d \in \underline{D}$.

Finally, let us show that

$$\downarrow^{\mathrm{p}}(-) : \begin{cases} D & \longrightarrow \langle \Downarrow^{\mathrm{p}} \underline{D}, \subseteq \rangle \\ d & \mapsto \downarrow^{\mathrm{p}} d \end{cases} \text{ and } \bigvee(-) : \begin{cases} \langle \Downarrow^{\mathrm{p}} \underline{D}, \subseteq \rangle & \longrightarrow D \\ X & \mapsto \bigvee X \end{cases}$$

are inverse maps of each other in $\mathscr{P}$ (for the sake of clarity, we will indicate by subscript the order over which the lubs are taken):

- they are maps of prime algebraic domains:
  - they are clearly monotone
  - for all $X \subseteq \underline{D}$ finitely bounded, $\downarrow^{\mathrm{p}} \left(\bigvee_{\leq} X\right) \subseteq \overbrace{\bigvee_{\subseteq} \downarrow^{\mathrm{p}}[X]}^{= \bigcup \Downarrow^{\mathrm{p}} X}$: if $\downarrow^{\mathrm{p}} \left(\bigvee_{\leq} X\right) \ni p \leq \bigvee_{\leq} X$, then since p is a complete prime, there exists $d \in X$ such that $p \leq d$. So

    $$p \in \downarrow^{\mathrm{p}} d \subseteq \bigcup_{d' \in X} \downarrow^{\mathrm{p}} d' = \bigcup \Downarrow^{\mathrm{p}} X$$

  - for all $\mathscr{X} \subseteq \Downarrow^{\mathrm{p}} \underline{D}$ finitely bounded, $\bigvee_{\leq} \left(\overbrace{\bigvee_{\subseteq} \mathscr{X}}^{= \bigcup \mathscr{X}}\right) \leq_D \bigvee_{\leq} \left(\bigvee_{\leq}[\mathscr{X}]\right)$: it suffices to show that

    $$\bigcup \mathscr{X} \text{ is bounded by } \bigvee_{\leq} \left(\bigvee_{\leq}[\mathscr{X}]\right)$$

    which is straightforward: for all $x \in \bigcup \mathscr{X}$, there exists $X \in \mathscr{X}$ such that

    $$x \in X \leq_D \bigvee_{\leq} X \leq \bigvee_{X' \in \mathscr{X}} \bigvee_{\leq} X' = \bigvee_{\leq} \left(\bigvee_{\leq}[\mathscr{X}]\right)$$

- $\bigvee({\downarrow^{\mathrm{p}}}(-)) = \mathrm{id}_{\underline{D}}$: that's the prime algebraicity property.
- ${\downarrow^{\mathrm{p}}}(\bigvee(-)) = \mathrm{id}_{\Downarrow^{\mathrm{p}}\,\underline{D}}$: for all ${\downarrow^{\mathrm{p}}}\, d \in {\Downarrow^{\mathrm{p}}}\, \underline{D}$:

$${\downarrow^{\mathrm{p}}}\Big(\bigvee {\downarrow^{\mathrm{p}}}\, d\Big) \overset{\substack{\text{prime algebraicity}\\ \downarrow}}{=} {\downarrow^{\mathrm{p}}}(d)$$

$\blacksquare$

Lemma A.4.5

1. If $f : E \to F$ is a map of event structures, then the direct image $f[-] : \mathcal{C}E \to \mathcal{C}F$ is a map of prime algebraic domains.
2. Deduce that the mapping $E \mapsto \langle \mathcal{C}E, \subseteq \rangle$ extends to an essentially surjective faithful functor $\mathcal{C}-$ from $\mathcal{E}$ to the category $\mathscr{P}$ of finitary prime algebraic domains. This functor is not full.

Proof

1. We easily check that $f[-] : \mathcal{C}E \to \mathcal{C}F$ is a map of prime algebraic domains:
   - the *monotonicity* stems from direct image preserving inclusion
   - if $X \subseteq \underline{C}$ is finitely bounded, as $\bigvee X = \bigcup X$ (cf. Lemma A.4.1), the result follows from the fact that the direct image of a union is the union of the direct images.
2. The faithfulness of the functor stems from the previous point. Showing that it is *essentially surjective* is tantamount to showing

   $$\forall D \in \mathscr{P},\ \exists E \in \mathcal{E};\ D \simeq \langle \mathcal{C}E, \subseteq \rangle$$

   which is a direct consequence of Lemma A.4.4.

   It is **not full**, insofar as not all maps in $\operatorname{Hom}_{\mathscr{P}}(\mathcal{C}E, \mathcal{C}F)$ (for every event structure E, F) are direct images of maps in $\operatorname{Hom}_{\mathcal{E}}(E, F)$. Indeed, consider:
   - $\underline{E} := \{a \neq b\}$ and $\operatorname{Con}_E := \mathcal{P}(\underline{E})$.
   - $\underline{F} := \{\star\}$
   - let $\widetilde{f} : \mathcal{C}E \to \mathcal{C}F$ be the constant function mapping every configuration of E to $\{\star\}$.

   Then $\widetilde{f}$ is monotone (as it is constant) and preserves the lubs of finitely bounded sets, therefore it is a map of prime algebraic domains. However, if $\widetilde{f}$ were to be the direct image of a map $f : E \longrightarrow F$, then we would necessarily have

   $$f : \begin{cases} E \longrightarrow F & \\ a \mapsto \star & \text{(as } \widetilde{f}(\{a\}) = \{\star\}) \\ b \mapsto \star & \text{(as } \widetilde{f}(\{b\}) = \{\star\}) \end{cases}$$

   but f is not a map of event structures, as it is not injective on the configuration $\{a, b\} \in \mathcal{C}E$.

■

B. Event Structures as Presheaves

B.1 Presheaves as cocompletion

Reminder

Notation B.1. *Let* C *be a locally small category. One denotes*

- *its **category of presheaves** by* $\widehat{\mathbf{C}} := [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$
- *its **Yoneda embedding** by* $\mathbf{y}_{\mathbf{C}} : \begin{cases} \mathbf{C} \to \widehat{\mathbf{C}} \\ C \mapsto \mathrm{Hom}_{\mathbf{C}}(-, C) \end{cases}$

Lemma 2 — Yoneda Lemma.

For every presheaf $P \in \widehat{\mathbf{C}}$, there is an isomorphism

$$\mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}_{\mathbb{C}}(C), P) \overset{\overset{\text{natural in } C \text{ and } P}{\downarrow}}{\cong} P(C)$$

NB The Yoneda lemma, albeit elementary, is fundamental: it is underpinning many categorical ideas, and will be used extensively thereafter. Emily Riehl even goes as far as to say that:

> “The Yoneda lemma is arguably the most important result in category theory, although it takes some time to explore the depths of the consequences of this simple statement. [Rie16]”

Vocabulary B.1 A covariant functor $F : \mathbf{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists $C \in \mathbf{C}$ such that $F \cong \mathrm{Hom}_{\mathbf{C}}(C, -)$. A pair $\langle C, \varphi \rangle$ is called a **representation** of F if $\varphi : \mathrm{Hom}_{\mathbf{C}}(C, -) \longrightarrow F$ is a natural isomorphism.

Dually, a presheaf $P : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable if there exists $C \in \mathbf{C}$ such that $P \cong \mathbf{y}_{\mathbf{C}}(C)$, and a representation of P is a pair $\langle C, \varphi \rangle$ such that $\varphi : \mathbf{y}_{\mathbf{C}}(C) \longrightarrow P$ is a natural isomorphism.

NB

- Representations of functors are unique up to unique isomorphism.
- By the (proof of the) Yoneda lemma, such a natural transformations φ are entirely determined by their value $\varphi_C(\mathrm{id}_C)$ at id_C.

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor.

Vocabulary B.2

- F is an **equivalence** if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $\mathrm{Id}\mathbf{C} \cong G \circ F$ and $F \circ G \cong \mathrm{Id}\mathbf{D}$.
- F is **essentially surjective** if for all $d \in \mathbf{D}$, there exists $c \in \mathbf{C}$ such that $F(a) \cong d$
- F is **faithful** if for any $f, g : a \longrightarrow b$ in $\mathbf{C}$, $Ff = Fg$ implies $f = g$
- F is **full** if for any $a, b \in \mathbf{C}$ and $g : Fa \longrightarrow Fb$ in $\mathbf{D}$, there exists $f : a \longrightarrow b$ such that $Ff = g$
- F is **fully faithful** if it is full and faithful.

Theorem B.1.1 If we assume the axiom of choice:

$$F \text{ is an equivalence } \iff F \text{ is fully faithful and essentially surjective}$$

Definition B.1.1 — The category of elements $\int P$ of a presheaf $P : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is the category whose

- **objects** are pairs $\langle C, x\rangle$, where $C \in \mathbf{C}$ and $x \in PC$
- **morphisms** $\langle C, x\rangle \longrightarrow \langle C, x'\rangle$ are morphisms $f : C \longrightarrow C'$ in $\mathbf{C}$ such that $Pf(x') = x$

NB

- $\int P$ is easily shown to be isomorphic to the coslice $* \downarrow P$, where $* \in \mathbf{Set}$ is the singleton
- $\int P$ is also denoted by $\int_{\mathbf{C}} P$: the coend notation alludes to the idea that $\int P$ ”unfolds”/”unpacks” P by taking the union of the $P(C)$’s, for $C \in \mathbf{C}$, while still remembering how P acts on the set elements via the morphisms of $\mathbf{C}$. This will be made more precise later.

Vocabulary B.3 — A category is said to be essentially small if it is equivalent to a small one.
Vocabulary B.4 — A functor is co/continuous if it preserves co/limits.
Vocabulary B.5 — An object C of a category $\mathbf{C}$ is said to be small if $\mathrm{Hom}_{\mathbf{C}}(C, -)$ is cocontinuous.

Proposition B.1.2 — The Hom functor is continuous in both arguments. If A is an object of a category $\mathbf{C}$ and $\lim_i B_i, \mathrm{colim}_i B'_i$ exist in $\mathbf{C}$:

- $\mathrm{Hom}_{\mathbf{C}}(A, \lim_i B_i) \cong \lim_i \mathrm{Hom}_{\mathbf{C}}(A, B_i)$
- $\mathrm{Hom}_{\mathbf{C}}(\mathrm{colim}_i B'_i, A) \cong \lim_i \mathrm{Hom}_{\mathbf{C}}(B'_i, A)$

NB The functor $\mathbf{y}_{\mathbf{C}}(A) := \mathrm{Hom}_{\mathbf{C}}(-, A)$ is in $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$, so it does preserve limits, as colimits in $\mathbf{C}$ are limits in $\mathbf{C}^{\mathrm{op}}$.

Definition B.1.2 — A dinatural transformation $\alpha : F \overset{\bullet}{\longrightarrow} G$ from a functor $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ to $G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ is given by a family of arrows $(\alpha_C : F(C,C) \longrightarrow G(C,C))_{C \in \mathbf{C}}$ such that for every morphism $f : C \longrightarrow C'$, the following hexagonal diagram commutes:

$$\begin{tikzcd}
& F(C',C) \arrow[dl, "F(f,C)"'] \arrow[dr, "F(C',f)"] & \\
F(C,C) \arrow[d, "\alpha_C"'] & & F(C',C') \arrow[d, "\alpha_{C'}"] \\
G(C,C) \arrow[dr, "G(C,f)"'] & & G(C',C') \arrow[dl, "G(f,C')"] \\
& G(C,C') &
\end{tikzcd}$$

Vocabulary B.6 — A wedge for $G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ is a dinatural transformation from a constant functor Δ_d, for $d \in \mathbf{D}$, to G.

NB

- By abuse of notation, a wedge $\Delta_d \overset{\bullet}{\longrightarrow} G$ may be denoted by $d \overset{\bullet}{\longrightarrow} G$ or simply referred to as d.
- Similarly to cones, wedges $\left\{\Delta_d \overset{\bullet}{\longrightarrow} G\right\}_{d\in\mathbf{D}}$ for G and morphisms $\phi : d \longrightarrow d'$ making the evident hexagonal diagrams commute form a category.

Definition B.1.3 — The end $\int_c G(c,c) \in \mathbf{D}$ of a functor $G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ is a terminal wedge for G.

NB Likewise, for a functor $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$, we have the dual notions of *cowedge* (dinatural transformation from F to a constant functor) and *coend* $\int^c F(c,c) \in \mathbf{D}$ (initial cowedge).

Proposition B.1.3 — Co/continuous functors preserve co/ends. *i.e.* if $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ has an end (resp. coend) and $H : \mathbf{D} \longrightarrow \mathbf{E}$ is continuous (resp. cocontinuous): $H\left(\int_c F(c,c)\right) \cong \int_c HF(c,c)$ (resp. $H\left(\int^c F(c,c)\right) \cong \int^c HF(c,c)$). In particular, for every $d \in \mathbf{D}$, $\mathrm{Hom}_{\mathbf{D}}\left(\int^c F(c,c), d\right) \cong \int_c \mathrm{Hom}_{\mathbf{D}}\left(F(c,c), d\right)$ and $\mathrm{Hom}_{\mathbf{D}}\left(d, \int_c F(c,c)\right) \cong \int_c \mathrm{Hom}_{\mathbf{D}}\left(d, F(c,c)\right)$.

Proposition B.1.4 — Fubini theorem for ends:. If $F : \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{E}^{\mathrm{op}} \times \mathbf{E}$ is a functor and the ends below exist, there are canonical isomorphisms:

$$\int_{(c,e)} F(c,c,e,e) \simeq \int_e \int_c F(c,c,e,e) \cong \int_c \int_e F(c,c,e,e)$$

Proposition B.1.5 — Natural transformations as Ends. If $F, G : \mathbb{C} \longrightarrow \mathbb{D}$ are two functors between (essentially) small categories:

$$\mathrm{Hom}_{\mathbb{D}^{\mathbb{C}}}\left(F, G\right) \cong \int_c \mathrm{Hom}_{\mathbb{D}}\left(Fc, Gc\right)$$

Definition B.1.4 — The tensor (also called copower) in a category $\mathbf{C}$ is, provided it exists, a functor

$$\cdot : \begin{cases} \mathbf{Set} \times \mathbf{C} \longrightarrow \mathbf{C} \\ (S, C) \mapsto S \cdot C \end{cases} \qquad \text{such that} \qquad \mathrm{Hom}_{\mathbf{C}}\left(S \cdot C, C'\right) \overset{\substack{\text{naturally in } S, C, C' \\ \downarrow}}{\cong} \mathrm{Hom}_{\mathbf{Set}}\left(S, \mathrm{Hom}_{\mathbb{C}}\left(C, C'\right)\right)$$

Dually: a **cotensor** (also called power) in $\mathbf{C}$ is, provided it exists, a functor

$$=^{-} : \begin{cases} \mathbf{Set} \times \mathbf{C} \longrightarrow \mathbf{C} \\ (S, C) \mapsto C^S \end{cases} \qquad \text{such that} \qquad \mathrm{Hom}_{\mathbf{C}}\left(C, C'^S\right) \overset{\substack{\text{naturally in } S, C, C' \\ \downarrow}}{\cong} \mathrm{Hom}_{\mathbf{Set}}\left(S, \mathrm{Hom}_{\mathbb{C}}\left(C, C'\right)\right)$$

NB Every locally small category that has co/products has a co/tensor by setting:

$$S^C := \prod_{s\in S} C \qquad\qquad S \cdot C := \coprod_{s \in S} C$$

In $\mathbf{C} := \mathrm{Set}$, we will take the tensor to be the cartesian product, and the cotensor to be the internal Hom.

Definition B.1.5 — An initial arrow from an object $D \in \mathbf{D}$ to a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is an initial object in the coslice category $D \downarrow F$, *i.e.* a pair $\langle C, \varphi \rangle$ where $C \in \mathbf{C}$, $\varphi : D \longrightarrow F(C)$ such that for all $C' \in \mathbf{C}$ and $f : D \longrightarrow F(C')$, there exists a unique $\mathbf{C}$-morphism $g : C \longrightarrow C'$ such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[dr, "\forall f"'] & F(C) \arrow[d, dotted, "F(g)"] & & C \arrow[d, dotted, "\exists!\, g"] \\ & F(C') & & C' \end{tikzcd}$$

Dually, a **terminal arrow** from F to D is a terminal object in $F \downarrow D$

Theorem B.1.6 — Characterisation of adjunctions ((Lan98) IV.1.2. (ii)).

Each adjunction $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ is completely determined by:

$$\begin{cases} \text{the functor } G : \mathbf{D} \longrightarrow \mathbf{C} \\ \text{for all } C \in \mathbf{C}\text{: an object } \underline{F}(C) \in \mathbf{D} \text{ and an initial arrow } \eta_C : C \longrightarrow G\underline{F}(C) \text{ from } C \text{ to } G \end{cases}$$

Then, the functor F is defined by $\underline{F}$ on objects and by $\eta_{C'} \circ f$ on arrows $f : C \longrightarrow C'$.

B.1.1 Category of elements

Lemma B.1.7 If $Q \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$:

$$\int Q \cong \mathbf{y}_{\mathbb{C}} \downarrow Q$$

Proof
This a direct corollary of the Yoneda lemma:

$$\begin{tikzcd} \boxed{\int Q} & & & & \boxed{\mathbf{y}_{\mathbb{C}} \downarrow Q} & \\ & & \text{Yoneda lemma} \arrow[d] & & & \\ \langle C, x \rangle \arrow[r, "f"] & \langle C', x' \rangle & \cong & & Q & \\ QC \ni x & x' \in QC' \arrow[l, mapsto, "Qf"'] & & \mathbf{y}_{\mathbb{C}}(C) \arrow[ur, "\varphi^x"] \arrow[rr, "f \circ -"'] & & \mathbf{y}_{\mathbb{C}}(C') \arrow[ul, "\varphi^{x'}"'] \end{tikzcd}$$

■

Equivalent diagrams

Every diagram $D : I \longrightarrow \mathbb{C}$ can be shown to be "equivalent" – in a sense that is made precise below – to a diagram D' that has the property (4.1.1). As a result, the freely added colimit of D in the free cocompletion will be taken to be the presheaf $P_{D'}$.

To get a sense of what is happening, set P to be a representable presheaf $\mathbf{y}_{\mathbb{C}}(C)$, for $C \in \mathbb{C}$. By the full- and faithfulness of the Yoneda embedding and Lemma B.1.7:

$$\int \mathbf{y}_{\mathbb{C}}(C) \cong \mathbb{C} \downarrow C$$

As a matter of fact, in order for D to satisfy the property (4.1.1): for each $C := D_i \in D[I]$, for each $\mathbb{C}$-morphism $f : X \longrightarrow C$ going into C, there should exist exactly one $j \in I$ such that $f : D_j \longrightarrow C \in \mathrm{Hom}_{D(I)}\left(D_j, D_i\right)$. This can fail in two ways:

(A) either there is no $j \in I$ **such that** $f \in \mathrm{Hom}_{D(I)}\left(D_j, D_i\right)$, as exemplified by the following diagram $D : I \longrightarrow \mathbb{C}$:

$$\begin{tikzcd}
 & \boxed{\mathbb{C}} & & & \boxed{D[I]} & \\
A \arrow[r, "g"] & C & & D_1 = A \arrow[rr, "g"] \arrow[rrd, "g"'] & & C = D_2 \\
B \arrow[ru, "f"'] & & & D_3 = A & & C = D_4
\end{tikzcd}$$

where there should be two indices j, j' such that $D_j = D_{j'} = B$ and $f \in \mathrm{Hom}_{\mathbb{C}}\left(D_j, D_2\right) \cap \mathrm{Hom}_{\mathbb{C}}\left(D_{j'}, D_4\right)$, which is not the case here. To fix this and determine the "equivalent" diagram $D' : I' \longrightarrow \mathbb{C}$ that has the property (4.1.1), one simply adds new objects in I that satisfy what we want:

$$\begin{tikzcd}
 & \boxed{D[I]} & & & & \boxed{D'[I']} & \\
D_1 = A \arrow[rr, "g"] \arrow[rrd, "g"'] & & C = D_2 & & D'_1 = A \arrow[r, "g"] \arrow[rd, "g"'] & C = D'_2 & B = D'_5 \arrow[l, "f"'] \\
D_3 = A & & C = D_4 & & D'_3 = A & C = D'_4 & B = D'_6 \arrow[l, "f"]
\end{tikzcd}$$

The presheaf $P_{D'}$ associated to D' and D is then given by:

$$P_{D'}(A) = \{a_1, a_2\} \qquad P_{D'}(B) = \{b_1, b_2\} \qquad P_{D'}(C) = \{b_1, b_2\}$$

$$P_{D'}f = \begin{cases} P_{D'}(C) \longrightarrow P_{D'}(B) & \\ c_i \mapsto b_i & \forall i \in \{1, 2\} \end{cases} \qquad P_{D'}g = \begin{cases} P_{D'}(C) \longrightarrow P_{D'}(A) & \\ c_i \mapsto a_1 & \forall i \in \{1, 2\} \end{cases}$$

(B) or there are two $j \neq j' \in I$ **such that** $D_j = D_{j'}$ **and** $f \in \mathrm{Hom}_{D(I)}\left(D_j, D_i\right) \cap \mathrm{Hom}_{D(I)}\left(D_{j'}, D_i\right)$:

for instance, consider the following diagram $D : I \longrightarrow \mathbb{C}$:

$$\boxed{\mathbb{C}} \qquad\qquad\qquad \boxed{D[I]}$$

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, "f" description] \arrow[r, bend right, "h"'] & C \end{tikzcd} \qquad \begin{tikzcd} D_1 = A \arrow[rr, shift left, "g"] \arrow[rr, shift right, "f"'] & & C = D_2 \\ D_3 = A \arrow[urr, shift left, "f"] \arrow[urr, shift right, "h"'] & & \end{tikzcd}$$

where there should be only one index j such that $D_j = A$ and $f \in \mathrm{Hom}_{\mathbb{C}}(D_j, D_2)$, which is not the case here. To determine the "equivalent" diagram $D' : I' \longrightarrow \mathbb{C}$, one "merges" the two redundant arrows into one:

$$\boxed{D[I]} \qquad\qquad\qquad \boxed{D'[I']}$$

$$\begin{tikzcd} D_1 = A \arrow[rr, shift left, "g"] \arrow[rr, shift right, "f"'] & & C = D_2 \\ D_3 = A \arrow[urr, shift left, "f"] \arrow[urr, shift right, "h"'] & & \end{tikzcd} \qquad \begin{tikzcd} D_1 = A \arrow[r, bend left, "g"] \arrow[r, "f" description] \arrow[r, bend right, "h"'] & C = D_2 \end{tikzcd}$$

The presheaf $P_{D'}$ is defined as:

$$P_{D'}(A) = \{a_1\} \qquad P_{D'}(C) = \{c_1\} \qquad P_{D'}f = P_{D'}g = P_{D'}h = \begin{cases} P_{D'}(C) \longrightarrow P_{D'}(A) \\ c_1 \mapsto a_1 \qquad \forall i \in \{1, 2\} \end{cases}$$

NB

- in a way, these two fixes **(A)** and **(B)** are operations of *rewriting system* of diagrams in $\mathbb{C}$, the normal forms of which are the diagrams satisfying the property (4.1.1).
- the two fixes in **(A)** and **(B)** are, in a way, reciprocal of each other: intuitively, a colimit Ω (at least in Set) of a diagram $(D_i)_{i \in I}$ can be thought of as being given by an algebraic structure: the underlying set being the disjoint union $\underline{\Omega} = \bigsqcup_i D_i$ of the D_i's, and the identities (equations) over Ω are given by the colimiting cocone conditions, identifying thereby some elements in $\underline{\Omega}$. Roughly, the **(A)** and **(B)** fixes consist in applying the following transformations:

Applying the (A) and (B) fixes

	Before the fix: $\underline{\Omega} \supseteq$	**Before the fix:** $E \supseteq$	**After the fix:** $\underline{\Omega} \supseteq$	**After the fix:** $E \supseteq$
(A)	$\{x\}$	$\emptyset$	$\{x, y\}$	$\{"x = y"\}$
(B)	$\{x, y\}$	$\{"x = y"\}$	$\{x\}$	$\emptyset$

B.1.2 Density formula/co-Yoneda lemma

A different take on the matter would be through the lens of coends. Indeed, we have the following expression of presheaves as coends over representables, referred to as the **co-Yoneda lemma** (or density formula):

Theorem B.1.8 — co-Yoneda lemma/density formula.. For every presheaf $P : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$:

$$P \cong \int^{c} Pc \times \mathbf{y}_{\mathbb{C}} c$$

This is a particular case of *tensor product of functors*: if $F \in \widehat{\mathbb{C}}$ and $G : \mathbb{C} \longrightarrow \mathbf{D}$ where $\mathbf{D}$ is cocomplete:

$$F \otimes_{\mathbb{C}} G := \int^{c} F(c) \cdot G(c)$$

The tensor product can be understood as follows: let's picture $\mathbb{C}$-objects as nonshiny Lego blocks, their corresponding representables as the shiny versions, and $\mathbf{D}$-objects as real-world physical objects. F, as a colimit of representables, is a shiny Lego construction/gluing. G is to be thought of as turning each nonshiny Lego block into a real-world $\mathbf{D}$-object. Then, $F \otimes_{\mathbb{C}} G$ is the real-world gluing where each shiny Lego block in F has been replaced by the image of the nonshiny corresponding Lego block by G.

The co-Yoneda lemma expresses the fact that

$$P \cong P \otimes_{\mathbb{C}} \mathbf{y}_{\mathbb{C}}$$

This matches the intuition: replacing each shiny Lego block in P by itself yields P again!

B.1.3 Kan Extensions

Proposition B.1.9 — Kan extensions as adjoints. An immediate corollary due to the definition is that $\alpha \mapsto \alpha_K \circ \eta$ yields an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^{\widetilde{\mathbf{C}}}} \left(\underset{K}{\mathrm{Lan}}(F), G \right) \overset{\overset{\text{natural in } G}{\downarrow}}{\cong} \mathrm{Hom}_{\mathbf{D}^{\mathbf{C}}} \left(F, G \circ K \right)$$

thus $\langle \mathrm{Lan}_K(F), -_K \circ \eta \rangle$ **represents the functor** $\mathrm{Hom}_{\mathbf{D}^{\mathbf{C}}} \left(F, - \circ K \right)$. Besides, by Theorem B.1.6: if every functor $F \in [\mathbf{C}, \mathbf{D}]$ has a left Kan extension, then $\mathrm{LanK} \dashv - \circ \mathrm{K}$

Lemma B.1.10 — Left adjoints preserve left Kan extensions. If $L : \mathbf{D} \longrightarrow \mathbf{E}$ **is a left adjoint** and the left Kan extension of $F : \mathbf{C} \longrightarrow \mathbb{D}$ along $K : \mathbf{C} \longrightarrow \widetilde{\mathbf{C}}$ exists, L preserves $\mathrm{Lan}_K(F)$, *i.e.* : $\langle L \circ \mathrm{Lan}_K(F), L\eta \rangle$ is the left Kan extension of LF along K. In particular:

$$L \circ \underset{K}{\mathrm{Lan}}(F) \cong \underset{K}{\mathrm{Lan}}(LF)$$

Proof
Assume we have an adjunction:

$$\mathrm{Hom}_{\mathbf{E}} \left(Ld, e \right) \cong \mathrm{Hom}_{\mathbf{D}} \left(d, Re \right) \quad \forall d \in \mathbf{D}, e \in \mathbf{E}$$

Then by applying that, for any functor $G : \widetilde{\mathbf{C}} \longrightarrow \mathbf{E}$, at every $d = \mathrm{Lan}_K(F)(\tilde{c})$ and $e = G(\tilde{c})$ yields:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{E}}\left(L \circ \underset{K}{\mathrm{Lan}}(F), G\right) &\cong \mathrm{Hom}_{\mathbf{D}}\left(\underset{K}{\mathrm{Lan}}(F), RG\right)\\ &\cong \mathrm{Hom}_{\mathbf{D}}\left(F, RGK\right)\\ &\cong \mathrm{Hom}_{\mathbf{D}}\left(F, RGK\right)\\ &\cong \mathrm{Hom}_{\mathbf{D}}\left(LF, GK\right)\end{aligned}$$

all of these being natural in G, so $L \circ \mathrm{Lan}_K(F) \cong \mathrm{Lan}_K(LF)$ as $\langle \mathrm{Lan}_K(LF), \eta\rangle$ represents $\mathrm{Hom}_{\mathbf{D}}(LF, - \circ K)$. The unit is obtained by setting $G := L \circ \mathrm{Lan}_K(F)$ and taking the image of $\mathrm{id}_{L\circ \mathrm{Lan}_K(F)} \in \mathrm{Hom}_{\mathbf{E}}(L \circ \mathrm{Lan}_K(F), L \circ \mathrm{Lan}_K(F))$, which yields $L\eta$. ■

Theorem B.1.11 — Existence of Kan extensions along a functor into a cocomplete category.

Let $\mathbb{C}$ be a **small category**, and $K : \mathbb{C} \longrightarrow \widetilde{\mathbf{C}}$, $F : \mathbb{C} \longrightarrow \mathbf{D}$ be functors.
If $\mathbf{D}$ is **cocomplete**, $\mathrm{Lan}_K(F)$ exists and can be defined, for all $\tilde{C} \in \widetilde{\mathbf{C}}$, as:

$$\underset{K}{\mathrm{Lan}}(F)(\tilde{C}) := \mathrm{colim}\left(K \downarrow \tilde{C} \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathbf{D}\right)$$

On top of that, if F is **fully faithful**, the natural transformation $\eta : F \longrightarrow \mathrm{Lan}_K(F) \circ K$ is an isomorphism.

Proof
The proof is quite technical and involved: it can be found in Theorem B.1.6 (X.3.Th1, p.237). ■

Theorem B.1.12 — Left Kan extensions as coends. Whenever the tensors and the coend appearing in the following formula exist, so do $\mathrm{Lan}_K(F)$, where $K \in [\mathbf{C}, \widetilde{\mathbf{C}}]$, $F \in [\mathbf{C}, \mathbf{D}]$, and there are natural (in K, F) isomorphisms:

$$\underset{K}{\mathrm{Lan}}(F) \cong \int^{c} \mathrm{Hom}_D(Kc, -) \cdot Fc$$

Proof
We have natural (in G) isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\tilde{\mathbf{C}}}}\left(\int^{c} \mathrm{Hom}_D(Kc, -) \cdot Fc, G\right) &\cong \int_{\tilde{c}} \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} \mathrm{Hom}_D(Kc, \tilde{c}) \cdot Fc, G(\tilde{c})\right)\\ &\cong \int_{\tilde{c}}\int_{c} \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} \mathrm{Hom}_D(Kc, \tilde{c}) \cdot Fc, G(\tilde{c})\right)\\ &\cong \int_{\tilde{c}}\int_{c} \mathrm{Hom}_{\mathbf{D}}\left(\mathrm{Hom}_D(Kc, \tilde{c}), \mathrm{Hom}_{\mathbf{D}}(Fc, G(\tilde{c}))\right)\\ &\cong \int_{c}\int_{\tilde{c}} \mathrm{Hom}_{\mathbf{D}}\left(\mathrm{Hom}_D(Kc, \tilde{c}), \mathrm{Hom}_{\mathbf{D}}(Fc, G(\tilde{c}))\right)\\ &\cong \int_{c} \mathrm{Hom}_{\mathbf{D}^{\tilde{\mathbf{C}}}}\left(\mathrm{Hom}_D(Kc, -), \mathrm{Hom}_{\mathbf{D}}(Fc, G(-))\right)\\ &\cong \int_{c} \mathrm{Hom}_{\mathbf{D}}\left(Fc, G(Kc)\right)\\ &\cong \mathrm{Hom}_{\mathbf{D}}\left(F, GK\right)\end{aligned}$$

Thus, as $\langle \mathrm{Lan}_K(F), \eta\rangle$ represents $\mathrm{Hom}_{\mathbf{D}}(F, - \circ K)$, the result follows. ■

Corollary B.1.13 — co-Yoneda Lemma/density formula. For every presheaf $P : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ over a small category $\mathbb{C}$:

$$P \cong \int^{c} Pc \times \mathbf{y}_{\mathbb{C}} c$$

Proof
By definition:

$$P \cong \underset{\mathrm{Id}\mathbb{C}}{\mathrm{Lan}}(P)$$

And as Set is tensored, the coend expression of the Kan extension holds:

$$P \cong \underset{\mathrm{Id}\mathbb{C}}{\mathrm{Lan}}(P) \cong \int^{c} \mathrm{Hom}_{D}\,(c, -) \times Pc \underset{\substack{\uparrow \\ \text{commutativity in } Set}}{\cong} \int^{c} Pc \times \mathrm{Hom}_{D}\,(c, -)$$

NB Similarly, we retrieve the Yoneda lemma from $F \cong \mathrm{Lan}_{\mathrm{Id}\mathbb{C}}(F)$, which may explain the name of the co-Yoneda lemma.

■

Theorem B.1.14 — The functor $\mathbf{y}_{\mathbb{C}}$ is the free cocompletion of a small category $\mathbb{C}$.

1. The category $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete.
2. For every cocomplete category $\mathbf{D}$ and functor $F : \mathbb{C} \to \mathbf{D}$ there is a unique (up to isomorphism) cocontinuous functor $\hat{F} : \widehat{\mathbb{C}} \to \mathbf{D}$ making the evident diagram commute up to natural isomorphism:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] \arrow[dr, phantom, "\cong"] & \mathbf{D} \\
\widehat{\mathbb{C}} \arrow[ur, dotted, "\hat{F}"'] &
\end{tikzcd}$$

Proof

1. The colimits can be taken pointwise, as Set is cocomplete:

$$(\mathrm{colim}_i\, P_i)(C) := \mathrm{colim}_i(P_i(C))$$

It is straightforward to check that this is well defined and satisfies the desired property.

2. By Theorem 4.1: as $\mathbb{C}$ is small and $\mathbf{D}$ is cocomplete, $\mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)$ exists; and as the Yoneda embedding is fully faithful, we know on top of that that the unit is thereof is an isomorphism. One can then set $\hat{F} := \mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)$.

■

NB **Freeness of the construction**: Apart from the analogy with extension of continuous functions from dense subspaces in topology (since $\mathbf{y}_{\mathbb{C}}[\mathbb{C}]$ is "co-dense" in $\widehat{\mathbb{C}}$, in that every presheaf is a colimit of representables), one may wonder if this construction being *free* can be made precise in a categorical sense: i.e. is the cocompletion functor a left adjoint of a forgetful functor U?

It turns out that exhibiting such a left adjoint is more nettlesome than it may seem at first sight. Indeed,

- if the statement were purely 1-categorical (i.e. if the triangle commuted on the nose and the Yoneda extension $\hat{F}$ were actually unique (not just up to isomorphism)), then one could think that there is no problem whatsoever, for
  - $(\mathbb{C}, \mathbf{y}_{\mathbb{C}})$ would be an *initial arrow* from $\mathbb{C}$ to $U : \mathbf{Cocomp} \to \mathbf{Cat}$ (i.e. an initial object in $\Delta_{\mathbb{C}} \downarrow U$) for each category $\mathbb{C}$ (where $\mathbf{Cocomp}$ is the category of cocomplete categories with cocontinuous functors)
  - which is enough to have an adjunction (cf. Theorem B.1.6:

$$(\hat{-}) \dashv U$$

  But we're not even in this situation here, as $\hat{F}$ is unique "up to isomorphism" (i.e. all the extensions of F that make the triangle commute up to natural isomorphism are isomorphic).
- One may then think that there are two possible workarounds:
  - to keep the commutativity and the uniqueness "on the nose", one may want to work with specified colimits, by considering the category $\mathbf{Cocomp}'$ of cocomplete categories equipped with a functor Colim that associates to each diagram (in a given cocomplete category) a particular colimit. The morphisms thereof would then be the cocontinuous functors that preserve the chosen colimits.
  - or one may settle for a 2-categorical statement: if $\mathbf{Cocomp}$ now denotes the 2-category of cocomplete categories (whose 1-arrows are cocontinuous functors and 2-arrows are natural transformations), and $\mathbf{Cat}$ the 2-category of (small) categories, one may venture that there is a 2-adjunction

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "(\hat{-})"] \arrow[r, phantom, "\perp"] & \mathbf{Cocomp} \arrow[l, bend left, "U"] \end{tikzcd}$$

  But we're up a creek without a paddle anyway, as there is a size issue in any case: if $\mathbb{C}$ is small, then $\widehat{\mathbb{C}}$ is not small anymore in general (so the category $\mathbf{Cocomp}$ we're considering can't be the category of small cocomplete categories). As a result: $U(\widehat{\mathbb{C}})$ (which should be an object of $\mathbf{Cat}$) is not a small category either!

Completeness: One may also wonder: « where do the limits come from (as the presheaf category is also complete), given that we have only added the free colimits? » Part of the reason why may be because of the following fact: every cocomplete category that has a small dense subcategory is complete (the dense subcategory here being the representables). There is a direct and elegant proof of this, by showing that the limit of a diagram $(\hat{P}_i)_i$ in $\widehat{\mathbb{C}}$ is nothing else than the colimit of the forgetful functor from the category of cones over $(\hat{P}_i)_i$ with summit an object of the dense subcategory.

Corollary B.1.15 — Every presheaf is a canonical colimit of representables For every $P \in \widehat{\mathbb{C}}$, where $\mathbb{C}$ is small:

$$P \cong \operatorname{colim}\left(\mathbf{y}_{\mathbb{C}} \downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}}\right)$$

Proof

Upon applying Theorem B.1.14 with $\mathbf{D} := \widehat{\mathbb{C}}, F := \mathbf{y}_{\mathbb{C}}$, it comes that

$$\operatorname*{Lan}_{\mathbf{y}_{\mathbb{C}}}(\mathbf{y}_{\mathbb{C}}) \cong \mathrm{Id}\widehat{\mathbb{C}}$$

by uniqueness (up to isomorphism) of $\hat{F}$.

Using the colimit expression of the Kan extension stemming from Theorem B.1.11 and taking the image at P yields the result. ■

B.2 Nerve construction: $\mathcal{E} \to \widehat{\mathbb{P}_+}$

In a 2008 article on the n-Category Café titled 'How I Learned to Love the Nerve Construction', Tom Leinster said:

> The nerve construction is inherent in the theory of categories.

And quite understandably: the nerve construction is an application of the Kan extension apparatus which unifies various parts of fields such as (higher) category theory, (higher) homotopy theory, algebraic topology, algebraic geometry, … among others. To quote what Urs Schreiber wrote on the the corresponding nLab entry:

> Pretty much every notion of category and higher category comes, or should come, with its canonical notion of simplicial nerve […]

And in our case, the nerve construction is precisely what will enable us to see event structures as presheaves over finite partial orders of events.

B.2.1 Nerve-Realisation paradigm

Let $F : \mathbb{C} \longrightarrow \mathbf{D}$ be a functor from a small category to locally small cocomplete one.

Proposition B.2.1

$$\operatorname*{Lan}_{\mathbf{y}_{\mathbb{C}}}(F) \dashv \mathrm{N}_F \cong \operatorname*{Lan}_{F}(\mathbf{y}_{\mathbb{C}})$$

Proof

As $\mathbf{D}$ is tensored and cocomplete, one can resort to the coend expression of the left Kan extension (cf. Theorem B.1.12):

- $\mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F) \dashv \mathrm{N}_F$:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}}\left(\operatorname*{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)(P), D\right) &\cong \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(c), P\right) \cdot Fc, D\right) \\
&\cong \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} Pc \cdot Fc, D\right) \\
&\cong \int_{c} \mathrm{Hom}_{\mathbf{D}}\left(Pc \cdot Fc, D\right) \\
&\cong \int_{c} \mathrm{Hom}_{\mathbf{Set}}\left(Pc, \mathrm{Hom}_{\mathbf{D}}\left(Fc, D\right)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Set}}\left(P, \underbrace{\mathrm{Hom}_{\mathbf{D}}\left(F(-), D\right)}_{:= \mathrm{N}_F(D)}\right)
\end{aligned}$$

- $\mathrm{N}_F \cong \mathrm{Lan}_F(\mathbf{y}_{\mathbb{C}})$:

$$\begin{aligned}
\underset{F}{\mathrm{Lan}}(\mathbf{y}_{\mathbb{C}})(d) &\cong \int^{c} \mathrm{Hom}_{\mathbf{D}}\,(F(c), d) \cdot \mathbf{y}_{\mathbb{C}}(c) \\
&\cong \int^{c} \underbrace{\mathrm{Hom}_{\mathbf{D}}\,(F(c), d)}_{:=\mathrm{N}_F(d)(c)} \times \mathbf{y}_{\mathbb{C}}(c) \\
&\cong \int^{c} \mathbf{y}_{\mathbb{C}}(c) \times \mathrm{N}_F(d)(c) \\
&\cong \mathrm{N}_F(d) && \text{(co-Yoneda lemma)}
\end{aligned}$$

■

B.2.2 Nerve of the inclusion of finite paths into event structures

Proposition B.2.2 Let $\mathbb{P}_+$ be the category of non-empty paths, and $\widehat{\mathbb{P}}'$ be the category of $\mathbb{P}$-presheaves A with $A\emptyset = \mathbb{1}$. We have an equivalence of categories $\widehat{\mathbb{P}}' \simeq \widehat{\mathbb{P}_+}$.

Proof
$\mathbb{P}_+$ is a full subcategory of $\mathbb{P}$, we denote by $\iota : \mathbb{P}_+ \hookrightarrow \mathbb{P}$ its fully faithful embedding into $\mathbb{P}$.

By restricting its domain, any presheaf of $\widehat{\mathbb{P}}'$ can be as presheaf over $\mathbb{P}_+$: we have a functor

$$\widehat{\mathbb{P}}' \xrightarrow{-\circ\iota} \mathbb{P}_+$$

Let's show that it is essentially surjective and fully faithful, which will be sufficient to get result, by Theorem B.1.1.

- $-\circ\iota$ **is essentially surjective**: indeed, any presheaf $A \in \widehat{\mathbb{P}_+}$ can be extended to a presheaf $A' \in \widehat{\mathbb{P}}'$ such that $A' \circ \iota = A$ by setting:
  - $A'(\emptyset) := \mathbb{1}$
  - $A'(\emptyset \xrightarrow{!_P} P) := A'(P) \xrightarrow{!_P} \mathbb{1}$ for all $P \in \mathbb{P}$
  - this extension clearly preserves the new identity $\mathrm{id}_\emptyset$, and we check that it still preserves composition:
    - $*$ for all $P \xrightarrow{f} Q$ in $\mathbb{P}$,

$$A'(\underbrace{\emptyset \xrightarrow{!_P} P \xrightarrow{f} Q}_{=\,\emptyset \xrightarrow{!_Q} Q}) = A'(Q) \xrightarrow{!_Q} \mathbb{1} = A'(Q) \xrightarrow{A'(f)} A'(P) \xrightarrow{!_P} \mathbb{1} = A'(f)\ ;\ A'(!_P)$$

    - $*$ and there is no morphism whose codomain is $\emptyset$

- $-\circ\iota$ **is fully faithful**: for all $A, B \in \widehat{\mathbb{P}}'$, we clearly have:

$$\mathrm{Hom}_{\mathbb{P}_+}\,(A\iota, B\iota) \cong \mathrm{Hom}_{\widehat{\mathbb{P}}'}\,(A, B)$$

  as any natural transformation $\phi \in \mathrm{Hom}_{\widehat{\mathbb{P}}'}\,(A, B)$ cannot but be equal to $\mathrm{id}_\emptyset$ at $\emptyset$, since $A(\emptyset) = B(\emptyset) = \mathbb{1}$.

■

NB In what follows, the category of paths $\mathbb{P}_+$ will be assumed to be small, as it is essentially small (we may as well consider a skeleton thereof).

The nerve of the inclusion functor $\mathbb{P}_+ \overset{I_+}{\hookrightarrow} \mathcal{E}$ enables us to regard event structures as presheaves over non-empty paths:

$$\begin{tikzcd}
\mathbb{P}_+ \arrow[r, hook, "I_+"] \arrow[d, hook, "\mathbf{y}_{\mathbb{P}_+}"'] & \mathcal{E} \arrow[dl, "{\mathrm{N}_{I_+} := E \mapsto \mathrm{Hom}_{\mathcal{E}}\left(I_+(-), E\right) \cong \mathrm{Lan}_{I_+}(\mathbf{y}_{\mathbb{P}_+})}"] \\
\widehat{\mathbb{P}_+} &
\end{tikzcd}$$

NB Note that we can't apply the theorem of existence of the Kan extension $\mathrm{Lan}_{I_+}(\mathbf{y}_{\mathbb{P}_+})$ here, as $\mathcal{E}$ is not cocomplete.

The question is: is N_{I_+} fully faithful? That is, do we have, for all $E, E' \in \mathcal{E}$:

$$\mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(\mathrm{N}_{I_+}(E), \mathrm{N}_{I_+}(E')\right) \overset{?}{\cong} \mathrm{Hom}_{\mathcal{E}}\left(E, E'\right)$$

It would be the case if I_+ had a right adjoint $G : \mathcal{E} \longrightarrow \mathbb{P}_+$ such that the counit of adjunction is an isomorphism. Indeed, then:

$$\begin{aligned}
\mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(\mathrm{N}_{I_+}(E), \mathrm{N}_{I_+}(E')\right) &:= \mathrm{Hom}_{\widehat{\mathbb{P}_+}}\Big(\mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(I_+(-), E\right), \mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(I_+(-), E'\right)\Big) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{P}_+}}\Big(\mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(-, GE\right), \mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(I_+(-), E'\right)\Big) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(I_+(GE), E'\right) && \text{(by the Yoneda lemma)} \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{P}_+}}\left(E, E'\right) && \text{(as the counit is an iso)}
\end{aligned}$$

But we can show that I_+ has *no right adjoint* unfortunately: if it were the case, then for all $E \in \mathcal{E}, P \in \mathbb{P}_+$:

$$\mathrm{Hom}_{\mathcal{E}}\left(I_+(P), E\right) \;\cong\; \overset{\substack{\text{always a finite set as both posets are finite} \\ \downarrow}}{\mathrm{Hom}_{\mathcal{E}}\left(P, GE\right)}$$

but we can certainly find an event structure such that the left hand side is not a finite set: for example by setting $E := \mathbb{N}$ (the set of natural number) and $\mathrm{Con}_E := \mathcal{P}_{\text{fin}}(\mathbb{N})$.

Another argument, more abstract, to show that I_+ has no right adjoint, is that otherwise, $\mathbb{P}_+$ would be said to be **a coreflective subcategory** of the ambient category $\mathcal{E}$. And corelfective subcategories can be shown to be closed under colimits which exist in the ambient category: that is, every diagram whose values are in $\mathbb{P}_+$ and that has a colimit in $\mathcal{E}$ actually has a colimit in $\mathbb{P}_+$. In our situation, it is clearly not the case, as $\mathcal{E}$ has an initial object (*i.e.* the empty diagram has a colimit in $\mathcal{E}$) but not $\mathbb{P}_+$.

So we are still at square one when it comes to proving the full- and faithfulness of N_{I_+}. Another lead to explore would be the following, if we had the existence of $\mathrm{Lan}_{I_+}(I_+)$ and a right adjoint $I_+ \dashv G$ (which is not the case, as mentioned above) such that the counit ϵ is an iso, then as left adjoints preserve Kan extensions:

$$\operatorname*{Lan}_{I_+}(I_+) = \operatorname*{Lan}_{I_+}(I_+ \circ \mathrm{Id}_{\mathbb{P}_+}) \cong I_+ \circ \underbrace{\operatorname*{Lan}_{I_+}(\mathrm{Id}_{\mathbb{P}_+})}_{\text{it can be shown that: } \cong\, G} \overset{\epsilon}{\cong} \mathrm{Id}_{\mathcal{E}}$$

If we expand the colimit-expression of the Kan extension written above, we precisely obtain the definition of $\mathbb{P}_+ \xrightarrow{I_+} \mathcal{E}$ **being a dense functor**: this condition will turn out to be sufficient to show the full- and faithfulness of the nerve.

B.3 Density of non-empty paths and full- and faithfulness of the nerve

Theorem B.3.1 — Density of non-empty paths in event structures.
The inclusion functor $\mathbb{P}_+ \overset{I_+}{\hookrightarrow} \mathcal{E}$ is dense.

Proof
Let $E \in \mathcal{E}$ be an event structure. We need to show that E is a canonical colimit of non-empty paths:

$$E \overset{?}{\cong} \operatorname{colim}\left(I_+ \downarrow E \xrightarrow{U} \mathbb{P}_+ \xrightarrow{I_+} \mathcal{E}\right)$$

Let $\langle E, \phi\rangle$ be the cocone under $I_+ \downarrow E \xrightarrow{U} \mathbb{P}_+ \xrightarrow{I_+} \mathcal{E}$ that we want to prove is colimiting:

$$\forall\, \langle P, f\rangle \in I_+ \downarrow E, \quad \phi_{\langle P,f\rangle} = f$$

and let $\langle E', \psi\rangle$ be another such cocone. Let us show that there exists a unique rigid map of cocones $r : E \longrightarrow E'$.

$$\begin{tikzcd}
 & E' & \\
 & E \arrow[u, dotted, "\exists!\, r"] & \\
\langle P, f\rangle \arrow[ur, "\phi_{\langle P,f\rangle} := f"] \arrow[uur, bend left, "\psi_{\langle P,f\rangle}"] \arrow[rr, "h"'] & & \langle Q, g\rangle \arrow[ul, "\phi_{\langle Q,g\rangle} := g"'] \arrow[uul, bend right, "\psi_{\langle Q,g\rangle}"']
\end{tikzcd}$$

Notation B.2. *For every configuration $x \in \mathcal{C}E$, we will denote by $\iota_x : x \longrightarrow \underline{E}$ the inclusion map.*

- **Unicity**: If we have two such maps $r, r' : E \longrightarrow E'$: for all $e \in \underline{E}$, as the following triangle commutes:

$$\begin{tikzcd}
 & E' \\
\langle \downarrow e, \iota_{\downarrow e}\rangle \arrow[ur, "\psi_{\langle \downarrow e, \iota_{\downarrow e}\rangle}"] \arrow[r, "\iota_{\downarrow e}"'] & E \arrow[u, bend left, "r"] \arrow[u, bend right, "r'"']
\end{tikzcd}$$

  which necessarily implies that

$$r(e) = r(\iota_{\downarrow e}(e)) \overset{\substack{\text{commutativity}\\ \downarrow}}{=} \psi_{\langle \downarrow e, \iota_{\downarrow e}\rangle}(e) \underset{\substack{\uparrow\\ \text{commutativity}}}{=} r'(\iota_{\downarrow e}(e)) = r'(e)$$

- **Existence**: We set

$$r := \begin{cases} \underline{E} & \longrightarrow \underline{E'} \\ e & \mapsto \psi_{\langle \downarrow e, \iota_{\downarrow e} \rangle}(e) \end{cases}$$

- **Cocone map (commutativity)**: For every $\langle P, f \rangle \in I_+ \downarrow E$, let us show that the following triangle commutes:

$$\begin{tikzcd}
 & E' \\
\langle P, f \rangle \arrow[ur, "\psi_{\langle P, f \rangle}"] \arrow[r, "f"'] & E \arrow[u, "r"']
\end{tikzcd}$$

  It amount to show that $\psi_{\langle P, f \rangle}(p) = rf(p)$, for every $p \in \underline{P}$.

  Indeed: since $f[\underline{P}] \in \mathcal{C}^o E$ and f is injective (it is locally injective and $\underline{P} \in \mathcal{C}^o P$), its corestriction $f^{|f[\underline{P}]}$ to $f[\underline{P}]$ is a bijection, and by Lemma A.2.5, $f' := {f^{|f[\underline{P}]}}^{-1}$ remains a rigid map. Its restriction $f'_{|\downarrow f(p)}$ too, and both the inner and outer triangles commute (cocone condition) in the following diagram:

$$\begin{tikzcd}
 & E' & \\
 & E \arrow[u, "r"] & \\
\langle P, f \rangle \arrow[ur, "f"] \arrow[uur, bend left, "\psi_{\langle P, f \rangle}"] & & \langle \downarrow f(p), \iota_{\downarrow f(p)} \rangle \arrow[ll, "f'_{|\downarrow f(p)}"'] \arrow[ul, hook, "\iota_{\downarrow f(p)}"'] \arrow[uul, bend right, "\psi_{\langle \downarrow f(p), \iota_{\downarrow f(p)} \rangle}"']
\end{tikzcd}$$

  Consequently:

$$\begin{aligned}
rf(p) &:= \psi_{\langle \downarrow f(p), \iota_{\downarrow f(p)} \rangle}(f(p)) \\
&= \psi_{\langle P, f \rangle} f'_{|\downarrow f(p)}(f(p)) && \text{(commutativity of the outer triangle)} \\
&= \psi_{\langle P, f \rangle}(f'_{|\downarrow f(p)} f(p)) \\
&= \psi_{\langle P, f \rangle}(p) && \text{(by definition of } f')
\end{aligned}$$

- **Rigid map**: The important point to note is that:

  Proposition If $x' \subseteq x \in \mathcal{C}E$,

$$\psi_{\langle x', \iota_{x'} \rangle} = {\psi_{\langle x, \iota_x \rangle}}_{|x'}$$

  This stems for the commutativity (cocone condition) of

$$\begin{tikzcd}
 & E' & \\
\langle x', \iota_{x'} \rangle \arrow[ur, hook, "\psi_{\langle x', \iota_{x'} \rangle}"] \arrow[rr, hook] & & \langle x, \iota_x \rangle \arrow[ul, hook', "\psi_{\langle x, \iota_x \rangle}"']
\end{tikzcd}$$

  Now, we can show that r is a rigid map of event structures:

- *Monotonicity*: if $e' \leq e \in \underline{E}$:

$$\overset{\text{by the previous proposition, as } \downarrow e' \subseteq \downarrow e \in \mathcal{C}E}{} \qquad r(e') := \psi_{\langle \downarrow e', \iota_{\downarrow e'} \rangle}(e') \overset{\downarrow}{=} \psi_{\langle \downarrow e, \iota_{\downarrow e} \rangle}(e') \underset{\underset{\text{monotonicity of } \psi_{\langle \downarrow e, \iota_{\downarrow e} \rangle}}{\uparrow}}{\subseteq} \psi_{\langle \downarrow e, \iota_{\downarrow e} \rangle}(e) := r(e)$$

- *Local injectivity*: likewise, the injectivity of r on $x \in \mathcal{C}^{\mathrm{o}}E$ comes for the injectivity of $\psi_{\langle x, \iota_x \rangle}$ on x

- *Preservation of configurations*: if $x \in\in \mathcal{C}^{\mathrm{o}}E$:

$$\overset{\text{by the previous proposition, as } \downarrow e \subseteq x \in \mathcal{C}^{\mathrm{o}}E \quad \forall e \in x}{} \qquad r[x] = \bigcup_{e \in x} \underbrace{\{r(e)\}}_{:= \psi_{\langle \downarrow e, \iota_{\downarrow e} \rangle}(e)} \quad \overset{\downarrow}{=} \quad \bigcup_{e \in x} \{\psi_{\langle x, \iota_x \rangle (e)}\} \quad = \quad \psi_{\langle x, \iota_x \rangle}[x] \underset{\underset{\text{as } \psi_{\langle x, \iota_x \rangle} \text{ is configuration-preserving}}{\uparrow}}{\in} \mathcal{C}^{\mathrm{o}}E$$

■

Theorem B.3.2 Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a **cocontinuous** functor, $G : \mathbf{C} \longrightarrow \mathbf{D}$ a functor and $\mathbf{A} \overset{i}{\hookrightarrow} \mathbf{C}$ a dense subcategory. There is a unique extension of every natural transformation $\alpha i : Fi \longrightarrow Gi$ to a natural transformation $\alpha : F \longrightarrow G$.

Dually: If $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, $G : \mathbf{C} \longrightarrow \mathbf{D}$ a **continuous** functor and $\mathbf{A} \overset{i}{\hookrightarrow} \mathbf{C}$ a co-dense subcategory, there is a unique extension of every natural transformation $\alpha i : Fi \longrightarrow Gi$ to a natural transformation $\alpha : F \longrightarrow G$.

Proof
The proof is as shown in the diagram below:

$$\begin{tikzcd}
i(a_1) \arrow[dd, "g"'] \arrow[dr, "\varphi_1"] & & & F(ia_1) \arrow[rrr, "\alpha_{ia_1}"] \arrow[dd, "Fg"'] \arrow[dr, "F\varphi_1"] & & & G(ia_1) \arrow[dl, "G\varphi_1"'] \arrow[dd, "Gg"] \\
& c \cong \mathrm{colim}\,\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) & & & F(c) \cong \mathrm{colim}\,F\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[r, dashed, "\exists!\ \alpha_c"] & G(c) & \\
i(a_2) \arrow[ur, "\varphi_2"'] & & & F(ia_2) \arrow[ur, "F\varphi_2"'] \arrow[rrr, "\alpha_{ia_2}"'] & & & G(ia_2) \arrow[ul, "G\varphi_2"'] \\
& & {} \arrow[r, mapsto, dotted, "{F\,,\,G}"] & {} & & & \\
i(a_1') \arrow[dr, "\varphi_1'"] & & & F(ia_1') \arrow[rrr, "\alpha_{ia_1'}"] \arrow[dr, "F\varphi_1'"] & & & G(ia_1') \arrow[dl, "G\varphi_1'"'] \\
& c' \cong \mathrm{colim}\,\big((i/c') \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[uuuu, "f"'] & & & F(c') \cong \mathrm{colim}\,F\big((i/c') \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[uuuu, "Ff"] \arrow[r, dashed, "\exists!\ \alpha_{c'}"] \arrow[uuuur, dashed, "\exists!"] & G(c') \arrow[uuuu, "Gf"'] & \\
i(a_2') \arrow[uu, "g'"] \arrow[ur, "\varphi_2'"'] & & & F(ia_2') \arrow[uu, "Fg'"] \arrow[ur, "F\varphi_2'"'] \arrow[rrr, "\alpha_{ia_2'}"'] & & & G(ia_2') \arrow[uu, "Gg'"'] \arrow[ul, "G\varphi_2'"']
\end{tikzcd}$$

For every $c \cong \mathrm{colim}\,\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big)$, $c' \cong \mathrm{colim}\,\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \in \mathbf{C}$, $f : c \longrightarrow c'$:

- there is a cocone from $\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \xrightarrow{F} \mathbf{D}$ to $G(c)$ (resp. from $\big((i/c') \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \xrightarrow{F} \mathbf{D}$ to $G(c')$). *Indeed:* the green subdiagram commutes (since $G\varphi_1 = G\varphi_2 \circ Gg$ (cocone property) and $Gg \circ \alpha_{ia_1} = \alpha_{ia_2} \circ Fg$ (naturality)) so there exists a unique corresponding cocone morphism $\alpha_c : F(c) \longrightarrow G(c)$ (resp. $\alpha'_c : F(c') \longrightarrow G(c')$), due to F being cocontinuous.

- the two cocones $\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \xrightarrow{F} \mathbf{D}$ to $G(c)$ obtained by post-composing by $\alpha_c \circ Ff$ and $Gf \circ \alpha_{c'}$ are equal. *Indeed:* the red subdiagram commutes since $G\varphi'_1 \circ \alpha_{ia'_1} = \alpha_{c'} \circ F\varphi'_1$ (colimit property of $F(c')$) and $\alpha_c \circ Ff \circ F\varphi'_1 = Gf \circ G\varphi'_1 \circ \alpha_{ia'_1}$ (colimit property of $F(c)$)) so there is a unique cocone morphism from $\big((i/c) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \xrightarrow{F} \mathbf{D}$ to $G(c)$ for these cocones, and the red square commutes: $Gf \circ \alpha_{c'} = \alpha_c \circ Ff$, which ends the proof.

Dually:

$$\begin{tikzcd}
i(a_1) \arrow[dd, "g"'] & & & F(ia_1) \arrow[rrr, "\alpha_{ia_1}"] \arrow[dd, "Fg"'] & & & G(ia_1) \arrow[dd, "Gg"] \\
& c \cong \lim\big((c\backslash i) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[ul, "\varphi_1"'] \arrow[dl, "\varphi_2"] & & & F(c) \arrow[ul, "F\varphi_1"'] \arrow[dl, "F\varphi_2"] \arrow[r, dashed, "\exists!\ \alpha_c"] & G(c) \cong \lim G\big((c\backslash i) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[ur, "G\varphi_1"] \arrow[dr, "G\varphi_2"] & \\
i(a_2) & & & F(ia_2) \arrow[rrr, "\alpha_{ia_2}"'] & & & G(ia_2) \\
& & {} \arrow[r, dotted, mapsto, "{F\,,\,G}"] & {} & & & \\
i(a'_1) & & & F(ia'_1) \arrow[rrr, "\alpha_{ia'_1}"] & & & G(ia'_1) \\
& c' \cong \lim\big((c'\backslash i) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[ul, "\varphi'_1"'] \arrow[dl, "\varphi'_2"] \arrow[uuuu, "f"] & & & F(c') \arrow[ul, "F\varphi'_1"'] \arrow[dl, "F\varphi'_2"] \arrow[uuuu, "Ff"] \arrow[uuuur, dashed, "\exists!"] \arrow[r, dashed, "\exists!\ \alpha_{c'}"] & G(c') \cong \lim G\big((c'\backslash i) \xrightarrow{U} \mathbf{A} \xrightarrow{i} \mathbf{C}\big) \arrow[ur, "G\varphi'_1"] \arrow[dr, "G\varphi'_2"] \arrow[uuuu, "Gf"'] & \\
i(a'_2) \arrow[uu, "g'"] & & & F(ia'_2) \arrow[uu, "Fg'"] \arrow[rrr, "\alpha_{ia'_2}"'] & & & G(ia'_2) \arrow[uu, "Gg'"']
\end{tikzcd}$$

■

NB The result *doesn't stand* in general when F is not cocontinuous, here are two counter-examples to the existence and unicity of the natural transformation extension in this case:

- **Counter-example to existence**: Let
  - $\mathbf{C} = \mathbf{D} := X \longrightarrow Y$ (category with two objects and one non-trivial morphism between them)
  - $\mathbf{A} := \mathbb{1}$ (the terminal category)
  - $i : \begin{cases} \mathbf{A} \hookrightarrow \mathbf{C} \\ * \mapsto Y \end{cases}$
  - $F := \mathrm{const}_Y \in [\mathbf{C}, \mathbf{D}]$ (constant functor sending all the $\mathbf{C}$-objects to Y)
  - $G := \mathrm{Id}\mathbf{C} \in [\mathbf{C}, \mathbf{D}]$

  then there is a natural transformation between $Fi : \begin{cases} \mathbb{1} \longrightarrow \mathbf{D} \\ * \mapsto Y \end{cases}$ and $Gi : \begin{cases} \mathbb{1} \longrightarrow \mathbf{D} \\ * \mapsto Y \end{cases}$ (the identity), but no natural transformation from F to G, as there is no morphism from $F(X) = Y$ and $G(X) = X$.
- **Counter-example to uniqueness**:

- As before, let's set $\mathbf{C} := X \longrightarrow Y,\ \mathbf{A} := \mathbb{1},\ i : \begin{cases} \mathbf{A} \hookrightarrow \mathbf{C} \\ * \mapsto Y \end{cases}$
- Let $\mathbf{D}$ be the category given by
  - two object X' and Y'
  - two non-trivial morphisms: $g : X' \longrightarrow Y'$ and $f : X' \longrightarrow X'$ such that

$$gf = g \qquad \text{and} \qquad f^2 = \mathrm{id}_{X'}$$

- $F = G : \begin{cases} \mathbf{C} \longrightarrow \mathbf{D} \\ X \mapsto X' \\ Y \mapsto Y' \end{cases}$

then there is a natural transformation between $Fi : \begin{cases} \mathbb{1} \longrightarrow \mathbf{D} \\ * \mapsto Y' \end{cases}$ and $Gi : \begin{cases} \mathbb{1} \longrightarrow \mathbf{D} \\ * \mapsto Y' \end{cases}$ (the identity), but two distinct natural transformations from F to G (the identity and the one whose component at X is f).

Application: Nerve functor fully faithful

Corollary B.3.3 If $\mathbb{C} \overset{i}{\hookrightarrow} \mathbf{D}$ is dense, the nerve functor

$$\mathrm{N}_i : \begin{cases} \mathbf{D} & \longrightarrow \widehat{\mathbb{C}} \\ d & \mapsto \mathrm{Hom}_{\mathbf{D}}\left(i(-), d\right) \end{cases}$$

is fully faithful.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook, "i"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] & \mathbf{D} \arrow[dl, "\mathrm{N}_i"] \\
\widehat{\mathbb{C}} &
\end{tikzcd}$$

Proof
We want to prove that for all $d, d' \in \mathbf{D}$:

$$\mathrm{Nat}\left(\underbrace{\mathrm{Hom}_{\mathbf{D}}\left(i(-), d\right)}_{\mathbf{y}_{\mathbf{D}}(d)i},\ \underbrace{\mathrm{Hom}_{\mathbf{D}}\left(i(-), d'\right)}_{\mathbf{y}_{\mathbf{D}}(d')i}\right) \cong \mathrm{Nat}(d,\ d') \overset{\substack{\text{Yoneda lemma} \\ \downarrow}}{\cong} \mathrm{Nat}\left(\underbrace{\mathrm{Hom}_{\mathbf{D}}\left(-, d\right)}_{\mathbf{y}_{\mathbf{D}}(d)},\ \underbrace{\mathrm{Hom}_{\mathbf{D}}\left(-, d'\right)}_{\mathbf{y}_{\mathbf{D}}(d')}\right)$$

The result follows by applying the dual version of the lemma with i^{op} (i post-composed with the op functor) and $F := \mathbf{y}_{\mathbf{D}}(d),\ G := \mathbf{y}_{\mathbf{D}}(d')$:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook, "i^{\mathrm{op}}\text{co-dense}"] & \mathbf{D}^{\mathrm{op}} \arrow[r, shift left, "\mathbf{y}_{\mathbf{D}}(d)"] \arrow[r, shift right, "\mathbf{y}_{\mathbf{D}}(d')\text{ continuous}"'] & \mathbf{Set}
\end{tikzcd}$$

■

Lemma B.3.4 The nerve functor of the embedding $I_+ : \mathbb{P}_+ \hookrightarrow \mathcal{E}$ is fully faithful.

Proof

This a direct consequence of Theorem B.3.1 and Corollary B.3.3. ■