4. Event Structures as Presheaves

4.1 Presheaves as cocompletion

Let $\mathbb{C}$ be a small category and $\mathbf{y}_{\mathbb{C}}$ its Yoneda embedding. The goal of this section is to show the following theorem and understand it intuitively:

Theorem 1 — The functor $\mathbf{y}_{\mathbb{C}}$ is the free cocompletion of $\mathbb{C}$.

For every cocomplete category $\mathbf{D}$ and functor $F : \mathbb{C} \to \mathbf{D}$ there is a unique (up to isomorphism) cocontinuous functor $\hat{F} : \widehat{\mathbb{C}} \to \mathbf{D}$ making the evident diagram commute up to natural isomorphism:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] \arrow[dr, phantom, "\cong"] & \mathbf{D} \\
\widehat{\mathbb{C}} \arrow[ur, dotted, "\hat{F}"'] &
\end{tikzcd}$$

A puzzling question may arise when considering the free cocompletion $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ of $\mathbb{C}$:

> What on earth does freely adding all its colimits to $\mathbb{C}$ have to do with presheaves? 4.1.1

4.1.1 Category of elements

A first approach to answer question 4.1.1 would be to investigate what happens when one adds, for each diagram in $\mathbb{C}$, its formal colimit, up until the resulting category $\widetilde{\mathbb{C}}$ ends up being cocomplete.

Example 4.1 — Adding a formal coproduct. For example, consider the simplest non-trivial example of colimit: the coproduct. Suppose that we have two distinguished objects $A, B \in \mathbb{C}$ and we want to define their formal coproduct – that will be denoted by Ω – in the category [1] $\widetilde{\mathbb{C}}$, so that the following universal property holds: for every functor $F : \mathbb{C} \longrightarrow \mathbf{D}$ such that $F(A), F(B)$ have a coproduct in $\mathbf{D}$, there exists a unique (up to isomorphism) functor $\hat{F} : \widetilde{\mathbb{C}} \to \mathbf{D}$ such that the evident triangle commutes:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook] & \mathbf{D} \\
\widetilde{\mathbb{C}} \arrow[ur, dotted, "\hat{F}"'] &
\end{tikzcd}$$

and $\hat{F}(\Omega) \cong F(A) + F(B)$

Then, defining $\Omega \in \widetilde{\mathbb{C}}$ amounts to describe, for all $C \in \mathbb{C}$, the morphisms:

- *going out of* Ω: this one is a cakewalk, we just use the universal property of the coproduct:

$$\mathrm{Hom}_{\widetilde{\mathbb{C}}}(\Omega, C) := \mathrm{Hom}_{\mathbb{C}}(A, C) \times \mathrm{Hom}_{\mathbb{C}}(B, C)$$

- *going into* Ω: By universal property of the coproduct, we do have a map

$$\mathrm{Hom}_{\widetilde{\mathbb{C}}}(C, A) + \mathrm{Hom}_{\widetilde{\mathbb{C}}}(C, B) \longrightarrow \mathrm{Hom}_{\widetilde{\mathbb{C}}}(C, \Omega)$$

Unfortunately, it may not be bijective at all in general (having a map into the coproduct is not tantamount to having a map into A and a map into B)! But here's the thing: if you want the previous universal property, you're bound to set:

$$\mathrm{Hom}_{\widetilde{\mathbb{C}}}\,(C,\Omega) := \mathrm{Hom}_{\mathbb{C}}\,(C,A) + \mathrm{Hom}_{\mathbb{C}}\,(C,B)$$

■

[1] $\mathbb{C}$ is a full subcategory of $\widetilde{\mathbb{C}}$ and $|\widetilde{\mathbb{C}}| := |\mathbb{C}| \cup \{\Omega\}$

Now, coming back to the general case of freely adding (in $\widehat{\mathbb{C}}$) the colimit Ω_D of any diagram[1] $D : I \longrightarrow \mathbb{C}$, we may be tempted to generalise from the previous example by setting, for all $C \in \mathbb{C}$:

$$\underbrace{\mathrm{Hom}_{\widehat{\mathbb{C}}}\,(\Omega_D, C) := \lim_{i\in I} \mathrm{Hom}_{\mathbb{C}}\,(D_i, C)}_{\uparrow\ \text{enforcing the continuity of } \mathbf{y}_{\mathbb{C}}(C) \text{ for all } C\in\mathbb{C}} \qquad \text{and} \qquad \underbrace{\mathrm{Hom}_{\widehat{\mathbb{C}}}\,(C, \Omega_D) := \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathbb{C}}\,(C, D_i)}_{\uparrow\ \text{making all } C\in\mathbb{C} \text{ small}}$$

But this approach is too naive: for instance, the two following diagrams are different (since their index categories are), but they ought to have the same colimit:

$$\begin{tikzcd} A \arrow[dr, "f"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$$

So we don't want to add two distinct formal colimits (as we would if we were to add all the Ω_D for every diagram D) for them! How is it even possible to keep track of all the diagrams that should have the same colimit, in such a way that we would add theirs only once? This is where presheaves come into play! Indeed, each presheaf can be associated to a diagram in $\mathbb{C} \cong \mathbf{y}_{\mathbb{C}}\mathbb{C}$

Indeed, consider for instance the following diagram in $\mathbb{C} :=$ $\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \arrow[dl, bend left, "h"] \\ C & \end{tikzcd}$:

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] \arrow[dr, "f"] & B \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

To go about adding its colimit P in $\widehat{\mathbb{C}}$, the trick is to rely on the isomorphism $\mathbb{C} \cong \mathbf{y}_{\mathbb{C}}[\mathbb{C}]$, so as to see the diagram in $\mathbf{y}_{\mathbb{C}}[\mathbb{C}] \subseteq [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] = \widehat{\mathbb{C}}$. Now, consider the colimiting cocone with summit P over the resulting diagram:

$$\begin{tikzcd} & P & \\ \mathbf{y}_{\mathbb{C}}A \arrow[ur, "\varphi_1^A"] \arrow[rr, shift left, "\mathbf{y}_{\mathbb{C}}g"] \arrow[rr, shift right, "\mathbf{y}_{\mathbb{C}}f"'] \arrow[drr, "\mathbf{y}_{\mathbb{C}}f"'] & & \mathbf{y}_{\mathbb{C}}B \arrow[ul, "\varphi_1^B"'] \\ \mathbf{y}_{\mathbb{C}}A \arrow[uur, bend left, "\varphi_2^A"] \arrow[rr, "\mathbf{y}_{\mathbb{C}}g"'] & & \mathbf{y}_{\mathbb{C}}B \arrow[uul, bend right, "\varphi_1^B"'] \end{tikzcd}$$

As it happens, we are in front of the comma category $\mathbf{y}_{\mathbb{C}} \downarrow P$. And it turns out that it is isomorphic to the category of elements of P:

[1] I is called the *index category* of the functor D

Lemma — B.1.7. If $Q \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$:

$$\int Q \cong \mathbf{y}_{\mathbb{C}} \downarrow Q$$

As a result, we have an explicit description of P: in the colimiting cocone,

- each natural transformation $\mathbf{y}_{\mathbb{C}} X \xrightarrow{\phi_i^X} P$ (where $X \in \{A, B\}, i \in \{1, 2\}$) corresponds to an element $x_i \in P(X)$
- for each morphism $\begin{tikzcd} & P & \\ \mathbf{y}_{\mathbb{C}} X \arrow[ur, "\phi_i^X"] \arrow[rr, "\mathbf{y}_{\mathbb{C}} h"'] & & \mathbf{y}_{\mathbb{C}} Y \arrow[ul, "\phi_j^Y"'] \end{tikzcd}$ (where $h \in \{f, g\}$), $Pf(y_j) = x_i$

And all the sets $\{P(C)\}_{C \in \mathbb{C}}$ and the action of P on the $\mathbb{C}$-morphisms are obtained in this way, by isomorphism with the category of elements of P (which unfolds everything there is to know about P: its action on sets and morphisms). So, in our example, P is given by:

$$P(A) = \{a_1, a_2\} \qquad P(B) = \{b_1, b_2\} \qquad P(C) = \emptyset$$

$$Pf = \begin{cases} P(B) \longrightarrow P(A) \\ b_i \mapsto a_1 \qquad \forall i \in \{1, 2\} \end{cases} \qquad Pg = \begin{cases} P(B) \longrightarrow P(A) \\ b_i \mapsto a_i \qquad \forall i \in \{1, 2\} \end{cases} \qquad Ph : P(C) = \emptyset \overset{\substack{\text{initial map in } \mathbf{Set} \\ \downarrow}}{\hookrightarrow} P(B)$$

So a presheaf has been associated to our original diagram, acting as its colimit. But there is a catch: everything goes well in the above example, but it might not in general! The highlighted sentence above Lemma B.1.7 is the fallacy of the argument: for a $\mathbb{C}$-diagram $D : I \longrightarrow \mathbb{C}$ whose colimit is denoted by $P_D \in \widehat{\mathbb{C}}$, it may not be the case in general that

Property 4.1.1

$$\mathbf{y}_{\mathbb{C}} D \downarrow P_D \cong \overbrace{\mathbf{y}_{\mathbb{C}} \downarrow P_D}^{\cong \int P_D}$$

But it can be shown that every diagram D is "equivalent" – in a sense that is made precise in the appendix B.1.1 – to a diagram D' that has this property. As such, the freely added colimit of D in $\widehat{\mathbb{C}}$ will be taken to be the presheaf $P_{D'}$. Any diagram $D : I \longrightarrow \mathbb{C}$ is "equivalent" to the diagram $\int (\operatorname{colim} \mathbf{y}_{\mathbb{C}} D) \xrightarrow{U} \mathbb{C}$, where U is the evident forgetful functor. Thus, it appears that $P_D \cong \operatorname{colim} \left(\underbrace{\int (\underbrace{\operatorname{colim} \mathbf{y}_{\mathbb{C}} D}_{P_D})}_{\cong \mathbf{y}_{\mathbb{C}} \downarrow P} \xrightarrow{U} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}} \right)$. In general:

Theorem — B.1.15 - Every presheaf is a canonical colimit of representables. For all $P \in \widehat{\mathbb{C}}$,

$$P \cong \operatorname{colim} \left(\mathbf{y}_{\mathbb{C}} \downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}} \right)$$

A different take on the matter would be through the lens of coends: for every presheaf $P \in \widehat{\mathbb{C}}$, $P \cong \int^c Pc \times \mathbf{y}_{\mathbb{C}} c$ (this is referred to as the *co-Yoneda lemma*, see the appendix B.1.2 for more details).

4.1.2 Kan Extensions

Kan extensions are very expressive universal constructions that enable us to extend functors along one another. The Kan extension of a functor F can be thought of as the best approximation of F

extending its domain to a larger category. Their ubiquity throughout mathematics led MacLane to state in [Lan98]:

> The notion of Kan extensions subsumes all the other fundamental concepts of category theory. [p.248]

The approach resorting to Kan extensions will enable us to see the problem at a higher level of generality, from which Theorem 4.1 will ensue, rather than tackling the issue hands-on.

Definition 4.1.1 — A left Kan extension of a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ along a functor $K : \mathbf{C} \longrightarrow \widetilde{\mathbf{C}}$ is a functor $\mathrm{Lan}_K(F) : \widetilde{\mathbf{C}} \longrightarrow \mathbf{D}$ and a natural transformation $\eta : F \longrightarrow \mathrm{Lan}_K(F) \circ K$ (called the *unit*) which is an initial arrow from $F \in [\mathbf{C}, \mathbf{D}]$ to $- \circ K : [\widetilde{\mathbf{C}}, \mathbf{D}] \longrightarrow [\mathbf{C}, \mathbf{D}]$

In other words: for any $G : \widetilde{\mathbf{C}} \longrightarrow \mathbf{D}$ and $\gamma : F \longrightarrow GK$, there exists a unique natural transformation $\alpha : \mathrm{Lan}_K(F) \longrightarrow G$ such that $\alpha_K \circ \eta = \gamma$:

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, Rightarrow, "\eta"'] & \mathbf{D} \\ & \widetilde{\mathbf{C}} \arrow[ur, "\mathrm{Lan}_K(F)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, Rightarrow, "\gamma"'] & \mathbf{D} \\ & \widetilde{\mathbf{C}} \arrow[ur, "G"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, bend right, "K"'] & {} \arrow[d, Rightarrow, bend right, "\eta"'] & \mathbf{D} \\ & \widetilde{\mathbf{C}} \arrow[ur, bend left, "\mathrm{Lan}_K(F)"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Downarrow\alpha"] & \end{tikzcd}$$

NB: Kan extensions are unique up to unique isomorphism, which is why we commonly use a definite article (*the* [left Kan extension of F along K]) to refer to them.

Theorem — B.1.11 - Existence of Kan extensions along a functor into a cocomplete category.

Let $\mathbb{C}$ be a **small** category, and $K : \mathbb{C} \longrightarrow \widetilde{\mathbf{C}}$, $F : \mathbb{C} \longrightarrow \mathbf{D}$ be functors.
If $\mathbf{D}$ is **cocomplete**, $\mathrm{Lan}_K(F)$ exists and can be defined, for all $\tilde{C} \in \widetilde{\mathbf{C}}$, as:

$$\underset{K}{\mathrm{Lan}}(F)(\tilde{C}) := \mathrm{colim}\left(K \downarrow \tilde{C} \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathbf{D}\right)$$

On top of that, if F is **fully faithful**, the natural transformation $\eta : F \longrightarrow \mathrm{Lan}_K(F) \circ K$ is an isomorphism.

With the machinery of Kan extensions, presheaves being colimits of representables and Theorem 4.1 are straightforward corollaries of Theorem B.1.11. On top of that, we can express Kan extensions as coends (Theorem B.1.12), which in turn implies the co-Yoneda lemma (Corollary B.1.13).

4.2 Nerve construction: $\mathcal{E} \to \widehat{\mathbb{P}_+}$

In a 2008 article on the n-Category Café titled 'How I Learned to Love the Nerve Construction', Tom Leinster said:

> The nerve construction is inherent in the theory of categories.

And quite understandably: the nerve construction is an application of the Kan extension apparatus which unifies various parts of fields such as (higher) category theory, (higher) homotopy theory, algebraic topology, algebraic geometry, . . . among others. To quote what Urs Schreiber wrote on the the corresponding nLab entry:

> Pretty much every notion of category and higher category comes, or should come, with its canonical notion of simplicial nerve [. . .]

And in our case, the nerve construction is precisely what will enable us to see event structures as presheaves over finite partial orders of events.

4.2.1 Nerve-Realisation paradigm

The general setting is the following:

Definition 4.2.1 — Nerve-Realisation paradigm
Let $F : \mathbb{C} \longrightarrow \mathbf{D}$ be a functor from a small category to locally small cocomplete one.

- The left Kan extension of F along $\mathbf{y}_{\mathbb{C}}$ (the existence of which is due to Theorem B.1.11) is referred to as **Yoneda extension** or **the realisation functor** of F
- It has a right adjoint $\mathrm{N}_F := \underbrace{D \mapsto \mathrm{Hom}_{\mathbf{D}}\,(F(-), D)}_{\text{denoted by } \mathrm{Hom}_{\mathbf{D}}(F(=),-)}$ called the **nerve of** F:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] & \mathbf{D} \arrow[dl, bend left, "\mathrm{N}_F"] \\ \widehat{\mathbb{C}} \arrow[ur, bend left, "\mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

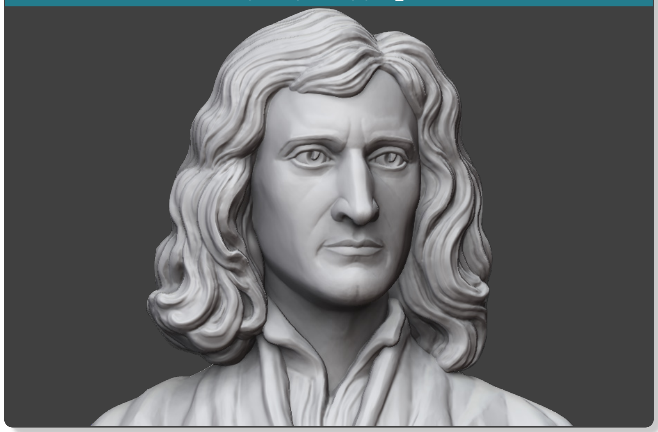
Proposition — B.2.1. With the above notations:

$$\underset{\mathbf{y}_{\mathbb{C}}}{\mathrm{Lan}}(F) \dashv \mathrm{N}_F \cong \underset{F}{\mathrm{Lan}}(\mathbf{y}_{\mathbb{C}})$$

To grasp an intuition for the nerve-realisation paradigm, let's bring back our Lego blocks example. You can think of $\mathbb{C}$-objects as being Lego blocks – and thus $\widehat{\mathbb{C}}$-objects as being Lego constructions due to $\widehat{\mathbb{C}}$ being the free cocompletion of $\mathbb{C}$ – and $\mathbb{D}$-objects as being real-world physical objects. Then:

- the functor F turns each Lego block into a real-world object (possibly a piece of a bigger object that would be colimit in $\mathbf{D}$).
- the realisation functor of F takes a Lego construction and replaces each of its Lego block by their real-world counterpart given by F
- the nerve functor of F associates to every real-world object the "closest matching" Lego construction for this object, by giving a way to probe the object with every Lego block. Indeed, $\mathrm{N}_F(d) := c \mapsto \mathrm{Hom}_{\mathbf{D}}\,(F(c), d)$ stores all the information with regard to "embedding" each Lego block c into the real-world object d

For example, our child may happen to have a bust of Newton in his bedroom (the bust being an object of $\mathbf{D}$ in our comparison), and may suddenly feel like making a Lego copy of it, thereby acting like the nerve:


Newton Bust $\in \mathbf{D}$


Newton Lego $\in \widehat{\mathbb{C}}$

Beyond the Lego analogy, $\mathbb{C}$ can really be seen as a category of basic "shapes" on the basis of which the realisation functor builds and the nerve "approximates" objects of $\mathbf{D}$. So we would like the nerve to be fully faithful, so that $\mathbf{D}$ is equivalent to a full subcategory of $\widehat{\mathbb{C}}$ (namely: the subcategory of presheaves isomorphic to an object of $\mathrm{N}_F[\mathbf{D}] \subseteq \widehat{\mathbb{C}}$), which would then mean that our Lego construction mechanism is sufficiently accurate to faithfully capture all the ways to transform a given real-world object into another one. Another way to see it is that if the nerve is fully faithful, $\mathrm{N}_F(d) \cong \mathrm{N}_F(d') \iff d \cong d'$ for all $d \in \mathbf{D}$, so we have injectivity on objects "up to isomorphism", that is: the category $\mathbf{D}$ is embeds into $\widehat{\mathbb{C}}$ "up to isomorphism".

What about event structures? As mentioned before, we do have a nerve construction in this setting as well.

4.2.2 Nerve of the inclusion of finite paths into event structures

Recall that the category of paths $\mathbb{P}$ is the category of finite posets seen as event structures (called *elementary event structures*, *path shapes* or simply *paths*) and rigid maps. As a matter of fact:

- a rigid map can be thought of as extending a path to another one.
- a presheaf A over $\mathbb{P}$ corresponds to a gluing of paths (as $\widehat{\mathbb{P}}$ is the free cocompletion of $\mathbb{P}$). As A is a colimit of representables and for all $P, Q \in \mathbb{P}$, $\mathbf{y}_{\mathbb{P}}(P)(Q) = \mathrm{Hom}_{\mathbb{P}}\,(Q, P)$ describes all the ways to embed Q into P, it follows that for all $P \in \mathbb{P}$, $A(P)$ can be thought of as the set of the states associated to the path P, that is: all the P-shaped computation paths that can be run by the process embodied by A.

But *all* the presheaves over $P \in \mathbb{P}$ are not relevant: for a given $A \in \widehat{\mathbb{P}}$, there should be only one computation path of shape $\emptyset$. So we ought to enforce $A(\emptyset) = \mathbb{1}$. That is what leads us to consider the category of presheaves $A \in \widehat{\mathbb{P}}$ such that $A(\emptyset) = \mathbb{1}$ (such presheaves are said to be *rooted*), denoted by $\widehat{\mathbb{P}}'$. And this category is equivalent to $\widehat{\mathbb{P}_+}$, where $\mathbb{P}_+$ is the category of non-empty paths (Proposition B.2.2).

As in [SW10], the nerve of the inclusion functor $\mathbb{P}_+ \overset{I_+}{\hookrightarrow} \mathcal{E}$ enables us to regard event structures as presheaves over non-empty paths: but is N_{I_+} fully faithful? To answer this, the discussion in appendix brings us to consider the *density* of the inclusion functor I_+.

4.3 Density of non-empty paths $\mathbb{P}_+$ in $\mathcal{E}$

Definition 4.3.1 A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is said to be **dense/codense** if every $\mathbf{C}$-object is a **canonical colimit/limit** of objects of $F[\mathbf{C}]$, *i.e.* for all $C \in \mathbf{C}$,

$$C \;\cong\; \mathrm{colim}\left(F \downarrow C \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D}\right) \quad / \quad C \;\cong\; \lim\left(C \downarrow F \xrightarrow{U} \mathbf{C} \xrightarrow{F} \mathbf{D}\right)$$

And we can show, in our case, that the inclusion of $\mathbb{P}_+$ in $\mathcal{E}$ is indeed dense:

Theorem — B.3.1. The inclusion functor $\mathbb{P}_+ \overset{I_+}{\hookrightarrow} \mathcal{E}$ is dense.

4.3.1 Sufficient condition for full- and faithfulness of the nerve

Finally, the fact that the density of a functor implies that its nerve is dense can be obtained as a corollary of the following theorem:

Theorem — B.3.2. If $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, $G : \mathbf{C} \longrightarrow \mathbf{D}$ a **continuous** functor and $\mathbf{A} \overset{i}{\hookrightarrow} \mathbf{C}$ a co-dense subcategory, there is a unique extension of every natural transformation $\alpha i : Fi \longrightarrow Gi$ to a natural transformation $\alpha : F \longrightarrow G$.

Corollary — B.3.3 If $\mathbb{C} \overset{i}{\hookrightarrow} \mathbf{D}$ is dense, the nerve functor N_i is fully faithful.

To reuse the Lego analogy, $\mathbb{C}$ being dense in $\mathbf{D}$ can be understood as the the Lego bricks being so small (let's say of atomic size!) that the Lego constructions are faithful enough to distinguish any two non-isomorphic real-world objects!

In the end, we straightforwardly deduce, due to the above corollary:

Lemma — B.3.4. The nerve functor for the embedding $I_+ : \mathbb{P}_+ \hookrightarrow \mathcal{E}$ is full and faithful.

4.4 Conclusion

On the whole, we first saw that how event structures, an operational model of concurrency, are related to special kinds of Scott domains: finitary prime algebraic ones. The configurations of an event structures form a finitary prime algebraic domain, and reciprocally, the set of complete primes of a finitary prime algebraic domain can be given the structure of an event structure. Prime algebraic domains are of crucial importance when it comes to dealing with denotational semantics, which would be the next step.

Second, the concept of free cocompletion was presented, and we sketched some reasons as to why it stems from presheaves, before hinting at a proof using the machinery of Kan extensions.

Lastly, we saw how Kan extensions are involved in the nerve realisation paradigm, and exhibited event structures as presheaves over non-empty paths via the nerve construction.

This was nothing but a tiny step toward studying event structures as presheaves: a lot remains to be done, the long-term objective being to investigate them through the lens of the theory of algebraic effects.

B. Event Structures as Presheaves

B.1 Presheaves as cocompletion

Reminder

Notation B.1. *Let* $\mathbf{C}$ *be a locally small category. One denotes*

- *its* ***category of presheaves*** *by* $\widehat{\mathbf{C}} := [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$
- *its* ***Yoneda embedding*** *by* $\mathbf{y_C} : \begin{cases} \mathbf{C} \to \widehat{\mathbf{C}} \\ C \mapsto \mathrm{Hom}_{\mathbf{C}}(-, C) \end{cases}$

Lemma 2 — Yoneda Lemma.

For every presheaf $P \in \widehat{\mathbf{C}}$, there is an isomorphism

$$\mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}_{\mathbb{C}}(C), P) \underset{\substack{\uparrow \\ \text{natural in } C \text{ and } P}}{\cong} P(C)$$

NB The Yoneda lemma, albeit elementary, is fundamental: it is underpinning many categorical ideas, and will be used extensively thereafter. Emily Riehl even goes as far as to say that:

> “The Yoneda lemma is arguably the most important result in category theory, although it takes some time to explore the depths of the consequences of this simple statement. [Rie16]”

Vocabulary B.1 A covariant functor $F : \mathbf{C} \longrightarrow \mathbf{Set}$ is **representable** if there exists $C \in \mathbf{C}$ such that $F \cong \mathrm{Hom}_{\mathbf{C}}(C, -)$. A pair $\langle C, \varphi \rangle$ is called a **representation** of F if $\varphi : \mathrm{Hom}_{\mathbf{C}}(C, -) \longrightarrow F$ is a natural isomorphism.

Dually, a presheaf $P : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable if there exists $C \in \mathbf{C}$ such that $P \cong \mathbf{y_C}(C)$, and a representation of P is a pair $\langle C, \varphi \rangle$ such that $\varphi : \mathbf{y_C}(C) \longrightarrow P$ is a natural isomorphism.

NB
- Representations of functors are unique up to unique isomorphism.
- By the (proof of the) Yoneda lemma, such a natural transformations φ are entirely determined by their value $\varphi_C(\mathrm{id}_C)$ at id_C.

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor.

Vocabulary B.2

- F is an **equivalence** if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $\mathrm{Id}\mathbf{C} \cong G \circ F$ and $F \circ G \cong \mathrm{Id}\mathbf{D}$.
- F is **essentially surjective** if for all $d \in \mathbf{D}$, there exists $c \in \mathbf{C}$ such that $F(a) \cong d$
- F is **faithful** if for any $f, g : a \longrightarrow b$ in $\mathbf{C}$, $Ff = Fg$ implies $f = g$
- F is **full** if for any $a, b \in \mathbf{C}$ and $g : Fa \longrightarrow Fb$ in $\mathbf{D}$, there exists $f : a \longrightarrow b$ such that $Ff = g$
- F is **fully faithful** if it is full and faithful.

Theorem B.1.1 If we assume the axiom of choice:

$$F \text{ is an equivalence} \iff F \text{ is fully faithful and essentially surjective}$$

Definition B.1.1 — The category of elements $\int P$ of a presheaf $P : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is the category whose

- **objects** are pairs $\langle C, x\rangle$, where $C \in \mathbf{C}$ and $x \in PC$
- **morphisms** $\langle C, x\rangle \longrightarrow \langle C, x'\rangle$ are morphisms $f : C \longrightarrow C'$ in $\mathbf{C}$ such that $Pf(x') = x$

NB
- $\int P$ is easily shown to be isomorphic to the coslice $* \downarrow P$, where $* \in \mathbf{Set}$ is the singleton
- $\int P$ is also denoted by $\int_{\mathbf{C}} P$: the coend notation alludes to the idea that $\int P$ "unfolds"/"unpacks" P by taking the union of the $P(C)$'s, for $C \in \mathbf{C}$, while still remembering how P acts on the set elements via the morphisms of $\mathbf{C}$. This will be made more precise later.

Vocabulary B.3 — A category is said to be essentially small if it is equivalent to a small one.
Vocabulary B.4 — A functor is co/continuous if it preserves co/limits.
Vocabulary B.5 — An object C of a category $\mathbf{C}$ is said to be small if $\mathrm{Hom}_{\mathbf{C}}(C, -)$ is cocontinuous.

Proposition B.1.2 — The Hom functor is continuous in both arguments. If A is an object of a category $\mathbf{C}$ and $\lim_i B_i, \operatorname{colim}_i B'_i$ exist in $\mathbf{C}$:

- $\mathrm{Hom}_{\mathbf{C}}(A, \lim_i B_i) \cong \lim_i \mathrm{Hom}_{\mathbf{C}}(A, B_i)$
- $\mathrm{Hom}_{\mathbf{C}}(\operatorname{colim}_i B'_i, A) \cong \lim_i \mathrm{Hom}_{\mathbf{C}}(B'_i, A)$

NB The functor $\mathbf{y}_{\mathbf{C}}(A) := \mathrm{Hom}_{\mathbf{C}}(-, A)$ is in $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$, so it does preserve limits, as colimits in $\mathbf{C}$ are limits in $\mathbf{C}^{\mathrm{op}}$.

Definition B.1.2 — A dinatural transformation $\alpha : F \overset{\bullet}{\longrightarrow} G$ from a functor $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ to $G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ is given by a family of arrows $(\alpha_C : F(C, C) \longrightarrow G(C, C))_{C \in \mathbf{C}}$ such that for every morphism $f : C \longrightarrow C'$, the following hexagonal diagram commutes:

$$\begin{tikzcd}
& F(C',C) \arrow[dl, "F(f,C)"'] \arrow[dr, "F(C',f)"] & \\
F(C,C) \arrow[d, "\alpha_C"'] & & F(C',C') \arrow[d, "\alpha_{C'}"] \\
G(C,C) \arrow[dr, "G(C,f)"'] & & G(C',C') \arrow[dl, "G(f,C')"] \\
& G(C,C') &
\end{tikzcd}$$

Vocabulary B.6 — A wedge for $G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ is a dinatural transformation from a constant functor Δ_d, for $d \in \mathbf{D}$, to G.

NB
- By abuse of notation, a wedge $\Delta_d \overset{\bullet}{\longrightarrow} G$ may be denoted by $d \overset{\bullet}{\longrightarrow} G$ or simply referred to as d.
- Similarly to cones, wedges $\left\{\Delta_d \overset{\bullet}{\longrightarrow} G\right\}_{d\in\mathbf{D}}$ for G and morphisms $\phi : d \longrightarrow d'$ making the evident hexagonal diagrams commute form a category.

Definition B.1.3 — The end $\int_c G(c,c) \in \mathbf{D}$ of a functor $G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ is a terminal wedge for G.

NB Likewise, for a functor $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$, we have the dual notions of *cowedge* (dinatural transformation from F to a constant functor) and *coend* $\int^c F(c,c) \in \mathbf{D}$ (initial cowedge).

Proposition B.1.3 — Co/continuous functors preserve co/ends. *i.e.* if $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ has an end (resp. coend) and $H : \mathbf{D} \longrightarrow \mathbf{E}$ is continuous (resp. cocontinuous): $H\left(\int_c F(c,c)\right) \cong \int_c HF(c,c)$ (resp. $H\left(\int^c F(c,c)\right) \cong \int^c HF(c,c)$). In particular, for every $d \in \mathbf{D}$, $\mathrm{Hom}_{\mathbf{D}}\left(\int^c F(c,c), d\right) \cong \int_c \mathrm{Hom}_{\mathbf{D}}\left(F(c,c), d\right)$ and $\mathrm{Hom}_{\mathbf{D}}\left(d, \int_c F(c,c)\right) \cong \int_c \mathrm{Hom}_{\mathbf{D}}\left(d, F(c,c)\right)$.

Proposition B.1.4 — Fubini theorem for ends:. If $F : \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \times \mathbf{E}^{\mathrm{op}} \times \mathbf{E}$ is a functor and the ends below exist, there are canonical isomorphisms:

$$\int_{(c,e)} F(c,c,e,e) \simeq \int_e \int_c F(c,c,e,e) \cong \int_c \int_e F(c,c,e,e)$$

Proposition B.1.5 — Natural transformations as Ends. If $F, G : \mathbb{C} \longrightarrow \mathbb{D}$ are two functors between (essentially) small categories:

$$\mathrm{Hom}_{\mathbb{D}^{\mathbb{C}}}\left(F, G\right) \cong \int_c \mathrm{Hom}_{\mathbb{D}}\left(Fc, Gc\right)$$

Definition B.1.4 — The tensor (also called copower) in a category $\mathbf{C}$ is, provided it exists, a functor

$$\cdot : \begin{cases} \mathbf{Set} \times \mathbf{C} \longrightarrow \mathbf{C} \\ (S, C) \mapsto S \cdot C \end{cases} \quad \text{such that} \quad \mathrm{Hom}_{\mathbf{C}}\left(S \cdot C, C'\right) \underset{\text{naturally in } S, C, C'}{\cong} \mathrm{Hom}_{\mathbf{Set}}\left(S, \mathrm{Hom}_{\mathbb{C}}\left(C, C'\right)\right)$$

Dually: a **cotensor** (also called power) in $\mathbf{C}$ is, provided it exists, a functor

$$=^{-} : \begin{cases} \mathbf{Set} \times \mathbf{C} \longrightarrow \mathbf{C} \\ (S, C) \mapsto C^S \end{cases} \quad \text{such that} \quad \mathrm{Hom}_{\mathbf{C}}\left(C, C'^S\right) \underset{\text{naturally in } S, C, C'}{\cong} \mathrm{Hom}_{\mathbf{Set}}\left(S, \mathrm{Hom}_{\mathbb{C}}\left(C, C'\right)\right)$$

NB Every locally small category that has co/products has a co/tensor by setting:

$$S^C := \prod_{s\in S} C \qquad\qquad S \cdot C := \coprod_{s\in S} C$$

In $\mathbf{C} := \mathbf{Set}$, we will take the tensor to be the cartesian product, and the cotensor to be the internal Hom.

Definition B.1.5 — An initial arrow from an object $D \in \mathbf{D}$ to a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is an initial object in the coslice category $D \downarrow F$, *i.e.* a pair $\langle C, \varphi \rangle$ where $C \in \mathbf{C}$, $\varphi : D \longrightarrow F(C)$ such that for all $C' \in \mathbf{C}$ and $f : D \longrightarrow F(C')$, there exists a unique $\mathbf{C}$-morphism $g : C \longrightarrow C'$ such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[dr, "\forall f"'] & F(C) \arrow[d, dotted, "F(g)"] \\ & F(C') \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d, dotted, "\exists ! \, g"] \\ C' \end{tikzcd}$$

Dually, a **terminal arrow** from F to D is a terminal object in $F \downarrow D$

Theorem B.1.6 — Characterisation of adjunctions ((Lan98) IV.1.2. (ii)).

Each adjunction $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ is completely determined by:

$$\begin{cases} \text{the functor } G : \mathbf{D} \longrightarrow \mathbf{C} \\ \text{for all } C \in \mathbf{C}\text{: an object } \underline{F}(C) \in \mathbf{D} \text{ and an initial arrow } \eta_C : C \longrightarrow G\underline{F}(C) \text{ from } C \text{ to } G \end{cases}$$

Then, the functor F is defined by $\underline{F}$ on objects and by $\eta_{C'} \circ f$ on arrows $f : C \longrightarrow C'$.

B.1.1 Category of elements

Lemma B.1.7 If $Q \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$:

$$\int Q \cong \mathbf{y}_{\mathbb{C}} \downarrow Q$$

Proof
This a direct corollary of the Yoneda lemma:

$$\begin{array}{ccc} \boxed{\int Q} & & \boxed{\mathbf{y}_{\mathbb{C}} \downarrow Q} \\ & \text{Yoneda lemma} & \\ & \downarrow & \\ \begin{tikzcd} \langle C, x \rangle \arrow[r, "f"] & \langle C', x' \rangle \\ QC \ni x & x' \in QC' \arrow[l, mapsto, "Qf"'] \end{tikzcd} & \cong & \begin{tikzcd} & Q & \\ \mathbf{y}_{\mathbb{C}}(C) \arrow[ur, "\varphi^{x}"] \arrow[rr, "f \circ -"'] & & \mathbf{y}_{\mathbb{C}}(C') \arrow[ul, "\varphi^{x'}"'] \end{tikzcd} \end{array}$$

■

Equivalent diagrams

Every diagram $D : I \longrightarrow \mathbb{C}$ can be shown to be "equivalent" – in a sense that is made precise below – to a diagram D' that has the property (4.1.1). As a result, the freely added colimit of D in the free cocompletion will be taken to be the presheaf $P_{D'}$.

To get a sense of what is happening, set P to be a representable presheaf $\mathbf{y}_{\mathbb{C}}(C)$, for $C \in \mathbb{C}$. By the full- and faithfulness of the Yoneda embedding and Lemma B.1.7:

$$\int \mathbf{y}_{\mathbb{C}}(C) \cong \mathbb{C} \downarrow C$$

As a matter of fact, in order for D to satisfy the property (4.1.1): for each $C := D_i \in D[I]$, for each $\mathbb{C}$-morphism $f : X \longrightarrow C$ going into C, there should exist exactly one $j \in I$ such that $f : D_j \longrightarrow C \in \mathrm{Hom}_{D(I)}(D_j, D_i)$. This can fail in two ways:

(A) either there is no $j \in I$ **such that** $f \in \mathrm{Hom}_{D(I)}(D_j, D_i)$, as exemplified by the following diagram $D : I \longrightarrow \mathbb{C}$:

$$\boxed{\mathbb{C}} \qquad\qquad \boxed{D[I]}$$

$$\begin{tikzcd} A \arrow[r, "g"] & C \\ B \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} D_1 = A \arrow[r, "g"] \arrow[dr, "g"'] & C = D_2 \\ D_3 = A & C = D_4 \end{tikzcd}$$

where there should be two indices j, j' such that $D_j = D_{j'} = B$ and $f \in \mathrm{Hom}_{\mathbb{C}}(D_j, D_2) \cap \mathrm{Hom}_{\mathbb{C}}\left(D_{j'}, D_4\right)$, which is not the case here. To fix this and determine the "equivalent" diagram $D' : I' \longrightarrow \mathbb{C}$ that has the property (4.1.1), one simply adds new objects in I that satisfy what we want:

$$\boxed{D[I]} \qquad\qquad \boxed{D'[I']}$$

$$\begin{tikzcd} D_1 = A \arrow[r, "g"] \arrow[dr, "g"'] & C = D_2 \\ D_3 = A & C = D_4 \end{tikzcd} \qquad \begin{tikzcd} D'_1 = A \arrow[r, "g"] \arrow[dr, "g"'] & C = D'_2 & B = D'_5 \arrow[l, "f"'] \\ D'_3 = A & C = D'_4 & B = D'_6 \arrow[l, "f"] \end{tikzcd}$$

The presheaf $P_{D'}$ associated to D' and D is then given by:

$$P_{D'}(A) = \{a_1, a_2\} \qquad P_{D'}(B) = \{b_1, b_2\} \qquad P_{D'}(C) = \{b_1, b_2\}$$

$$P_{D'}f = \begin{cases} P_{D'}(C) \longrightarrow P_{D'}(B) \\ c_i \mapsto b_i \qquad \forall i \in \{1,2\} \end{cases} \qquad P_{D'}g = \begin{cases} P_{D'}(C) \longrightarrow P_{D'}(A) \\ c_i \mapsto a_1 \qquad \forall i \in \{1,2\} \end{cases}$$

(B) or there are two $j \neq j' \in I$ **such that** $D_j = D_{j'}$ **and** $f \in \mathrm{Hom}_{D(I)}(D_j, D_i) \cap \mathrm{Hom}_{D(I)}\left(D_{j'}, D_i\right)$:

for instance, consider the following diagram $D : I \longrightarrow \mathbb{C}$:

$$\boxed{\mathbb{C}} \qquad\qquad\qquad \boxed{D[I]}$$

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, "f"] \arrow[r, bend right, "h"'] & C \end{tikzcd} \qquad \begin{tikzcd} D_1 = A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & C = D_2 \\ D_3 = A \arrow[ur, shift left, "f"] \arrow[ur, shift right, "h"'] & \end{tikzcd}$$

where there should be only one index j such that $D_j = A$ and $f \in \mathrm{Hom}_{\mathbb{C}}(D_j, D_2)$, which is not the case here. To determine the "equivalent" diagram $D' : I' \longrightarrow \mathbb{C}$, one "merges" the two redundant arrows into one:

$$\boxed{D[I]} \qquad\qquad\qquad \boxed{D'[I']}$$

$$\begin{tikzcd} D_1 = A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & C = D_2 \\ D_3 = A \arrow[ur, shift left, "f"] \arrow[ur, shift right, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} D_1 = A \arrow[r, bend left, "g"] \arrow[r, "f"] \arrow[r, bend right, "h"'] & C = D_2 \end{tikzcd}$$

The presheaf $P_{D'}$ is defined as:

$$P_{D'}(A) = \{a_1\} \qquad P_{D'}(C) = \{c_1\} \qquad P_{D'}f = P_{D'}g = P_{D'}h = \begin{cases} P_{D'}(C) \longrightarrow P_{D'}(A) \\ c_1 \mapsto a_1 \qquad \forall i \in \{1, 2\} \end{cases}$$

NB

- in a way, these two fixes **(A)** and **(B)** are operations of *rewriting system* of diagrams in $\mathbb{C}$, the normal forms of which are the diagrams satisfying the property (4.1.1).
- the two fixes in **(A)** and **(B)** are, in a way, reciprocal of each other: intuitively, a colimit Ω (at least in **Set**) of a diagram $(D_i)_{i \in I}$ can be thought of as being given by an algebraic structure: the underlying set being the disjoint union $\underline{\Omega} = \bigsqcup_i D_i$ of the D_i's, and the identities (equations) over Ω are given by the colimiting cocone conditions, identifying thereby some elements in $\underline{\Omega}$. Roughly, the **(A)** and **(B)** fixes consist in applying the following transformations:

Applying the (A) and (B) fixes				
	Before the fix: $\underline{\Omega} \supseteq$	**Before the fix:** $E \supseteq$	**After the fix:** $\underline{\Omega} \supseteq$	**After the fix:** $E \supseteq$
(A)	$\{x\}$	$\emptyset$	$\{x, y\}$	$\{"x = y"\}$
(B)	$\{x, y\}$	$\{"x = y"\}$	$\{x\}$	$\emptyset$

B.1.2 Density formula/co-Yoneda lemma

A different take on the matter would be through the lens of coends. Indeed, we have the following expression of presheaves as coends over representables, referred to as the **co-Yoneda lemma** (or density formula):

Theorem B.1.8 — co-Yoneda lemma/density formula.. For every presheaf $P : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$:

$$P \cong \int^{c} Pc \times \mathbf{y}_{\mathbb{C}} c$$

This is a particular case of *tensor product of functors*: if $F \in \widehat{\mathbb{C}}$ and $G : \mathbb{C} \longrightarrow \mathbf{D}$ where $\mathbf{D}$ is cocomplete:

$$F \otimes_{\mathbb{C}} G := \int^{c} F(c) \cdot G(c)$$

The tensor product can be understood as follows: let's picture $\mathbb{C}$-objects as nonshiny Lego blocks, their corresponding representables as the shiny versions, and $\mathbf{D}$-objects as real-world physical objects. F, as a colimit of representables, is a shiny Lego construction/gluing. G is to be thought of as turning each nonshiny Lego block into a real-world $\mathbf{D}$-object. Then, $F \otimes_{\mathbb{C}} G$ is the real-world gluing where each shiny Lego block in F has been replaced by the image of the nonshiny corresponding Lego block by G.

The co-Yoneda lemma expresses the fact that

$$P \;\cong\; P \otimes_{\mathbb{C}} \mathbf{y}_{\mathbb{C}}$$

This matches the intuition: replacing each shiny Lego block in P by itself yields P again!

B.1.3 Kan Extensions

Proposition B.1.9 — Kan extensions as adjoints. An immediate corollary due to the definition is that $\alpha \mapsto \alpha_K \circ \eta$ yields an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^{\tilde{\mathbf{C}}}} \left(\operatorname*{Lan}_{K}(F), G \right) \overset{\text{natural in } G}{\overset{\downarrow}{\cong}} \mathrm{Hom}_{\mathbf{D}^{\mathbf{C}}} \left(F, G \circ K \right)$$

thus $\langle \mathrm{Lan}_K(F), -_K \circ \eta \rangle$ **represents the functor** $\mathrm{Hom}_{\mathbf{D}^{\mathbf{C}}} \left(F, - \circ K \right)$. Besides, by Theorem B.1.6: if every functor $F \in [\mathbf{C}, \mathbf{D}]$ has a left Kan extension, then $\mathrm{LanK} \dashv - \circ \mathrm{K}$

Lemma B.1.10 — Left adjoints preserve left Kan extensions. If $L : \mathbf{D} \longrightarrow \mathbf{E}$ **is a left adjoint** and the left Kan extension of $F : \mathbf{C} \longrightarrow \mathbb{D}$ along $K : \mathbf{C} \longrightarrow \widetilde{\mathbf{C}}$ exists, L preserves $\mathrm{Lan}_K(F)$, *i.e.* : $\langle L \circ \mathrm{Lan}_K(F), L\eta \rangle$ is the left Kan extension of LF along K. In particular:

$$L \circ \operatorname*{Lan}_{K}(F) \cong \operatorname*{Lan}_{K}(LF)$$

Proof
Assume we have an adjunction:

$$\mathrm{Hom}_{\mathbf{E}} \left(Ld, e \right) \cong \mathrm{Hom}_{\mathbf{D}} \left(d, Re \right) \quad \forall d \in \mathbf{D}, e \in \mathbf{E}$$

Then by applying that, for any functor $G : \widetilde{\mathbf{C}} \longrightarrow \mathbf{E}$, at every $d = \mathrm{Lan}_K(F)(\tilde{c})$ and $e = G(\tilde{c})$ yields:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{E}}\left(L \circ \underset{K}{\mathrm{Lan}}(F), G\right) &\cong \mathrm{Hom}_{\mathbf{D}}\left(\underset{K}{\mathrm{Lan}}(F), RG\right) \\
&\cong \mathrm{Hom}_{\mathbf{D}}\left(F, RGK\right) \\
&\cong \mathrm{Hom}_{\mathbf{D}}\left(F, RGK\right) \\
&\cong \mathrm{Hom}_{\mathbf{D}}\left(LF, GK\right)
\end{aligned}$$

all of these being natural in G, so $L \circ \mathrm{Lan}_K(F) \cong \mathrm{Lan}_K(LF)$ as $\langle \mathrm{Lan}_K(LF), \eta\rangle$ represents $\mathrm{Hom}_{\mathbf{D}}\left(LF, - \circ K\right)$. The unit is obtained by setting $G := L \circ \mathrm{Lan}_K(F)$ and taking the image of $\mathrm{id}_{L \circ \mathrm{Lan}_K(F)} \in \mathrm{Hom}_{\mathbf{E}}\left(L \circ \mathrm{Lan}_K(F), L \circ \mathrm{Lan}_K(F)\right)$, which yields $L\eta$. ■

Theorem B.1.11 — Existence of Kan extensions along a functor into a cocomplete category.

Let $\mathbb{C}$ be a **small category**, and $K : \mathbb{C} \longrightarrow \widetilde{\mathbf{C}}$, $F : \mathbb{C} \longrightarrow \mathbf{D}$ be functors.
If $\mathbf{D}$ is **cocomplete**, $\mathrm{Lan}_K(F)$ exists and can be defined, for all $\tilde{C} \in \widetilde{\mathbf{C}}$, as:

$$\underset{K}{\mathrm{Lan}}(F)(\tilde{C}) := \mathrm{colim}\left(K \downarrow \tilde{C} \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathbf{D}\right)$$

On top of that, if F is **fully faithful**, the natural transformation $\eta : F \longrightarrow \mathrm{Lan}_K(F) \circ K$ is an isomorphism.

Proof
The proof is quite technical and involved: it can be found in Theorem B.1.6 (X.3.Th1, p.237). ■

Theorem B.1.12 — Left Kan extensions as coends. Whenever the tensors and the coend appearing in the following formula exist, so do $\mathrm{Lan}_K(F)$, where $K \in [\mathbf{C}, \widetilde{\mathbf{C}}]$, $F \in [\mathbf{C}, \mathbf{D}]$, and there are natural (in K, F) isomorphisms:

$$\underset{K}{\mathrm{Lan}}(F) \cong \int^{c} \mathrm{Hom}_D\left(Kc, -\right) \cdot Fc$$

Proof
We have natural (in G) isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^{\tilde{\mathbf{C}}}}\left(\int^{c} \mathrm{Hom}_D\left(Kc, -\right) \cdot Fc, G\right) &\cong \int_{\tilde{c}} \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} \mathrm{Hom}_D\left(Kc, \tilde{c}\right) \cdot Fc, G(\tilde{c})\right) \\
&\cong \int_{\tilde{c}}\int_{c} \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} \mathrm{Hom}_D\left(Kc, \tilde{c}\right) \cdot Fc, G(\tilde{c})\right) \\
&\cong \int_{\tilde{c}}\int_{c} \mathrm{Hom}_{\mathbf{D}}\left(\mathrm{Hom}_D\left(Kc, \tilde{c}\right), \mathrm{Hom}_{\mathbf{D}}\left(Fc, G(\tilde{c})\right)\right) \\
&\cong \int_{c}\int_{\tilde{c}} \mathrm{Hom}_{\mathbf{D}}\left(\mathrm{Hom}_D\left(Kc, \tilde{c}\right), \mathrm{Hom}_{\mathbf{D}}\left(Fc, G(\tilde{c})\right)\right) \\
&\cong \int_{c} \mathrm{Hom}_{\mathbf{D}^{\tilde{\mathbf{C}}}}\left(\mathrm{Hom}_D\left(Kc, -\right), \mathrm{Hom}_{\mathbf{D}}\left(Fc, G(-)\right)\right) \\
&\cong \int_{c} \mathrm{Hom}_{\mathbf{D}}\left(Fc, G(Kc)\right) \\
&\cong \mathrm{Hom}_{\mathbf{D}}\left(F, GK\right)
\end{aligned}$$

Thus, as $\langle \mathrm{Lan}_K(F), \eta\rangle$ represents $\mathrm{Hom}_{\mathbf{D}}\left(F, - \circ K\right)$, the result follows. ■

Corollary B.1.13 — co-Yoneda Lemma/density formula. For every presheaf $P : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ over a small category $\mathbb{C}$:

$$P \cong \int^{c} Pc \times \mathbf{y}_{\mathbb{C}} c$$

Proof
By definition:

$$P \cong \operatorname*{Lan}_{\mathrm{Id}\mathbb{C}}(P)$$

And as $\mathbf{Set}$ is tensored, the coend expression of the Kan extension holds:

$$P \cong \operatorname*{Lan}_{\mathrm{Id}\mathbb{C}}(P) \cong \int^{c} \mathrm{Hom}_{D}\,(c, -) \times Pc \underset{\substack{\uparrow \\ \text{commutativity in } Set}}{\cong} \int^{c} Pc \times \mathrm{Hom}_{D}\,(c, -)$$

NB: Similarly, we retrieve the Yoneda lemma from $F \cong \mathrm{Lan}_{\mathrm{Id}\mathbb{C}}(F)$, which may explain the name of the co-Yoneda lemma.

■

Theorem B.1.14 — The functor $\mathbf{y}_{\mathbb{C}}$ is the free cocompletion of a small category $\mathbb{C}$.

1. The category $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete.
2. For every cocomplete category $\mathbf{D}$ and functor $F : \mathbb{C} \to \mathbf{D}$ there is a unique (up to isomorphism) cocontinuous functor $\hat{F} : \widehat{\mathbb{C}} \to \mathbf{D}$ making the evident diagram commute up to natural isomorphism:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, hook, "\mathbf{y}_{\mathbb{C}}"'] \arrow[dr, phantom, "\cong"] & \mathbf{D} \\
\widehat{\mathbb{C}} \arrow[ur, dotted, "\hat{F}"'] &
\end{tikzcd}$$

Proof

1. The colimits can be taken pointwise, as $\mathbf{Set}$ is cocomplete:

$$(\mathrm{colim}_i\, P_i)(C) := \mathrm{colim}_i(P_i(C))$$

It is straightforward to check that this is well defined and satisfies the desired property.

2. By Theorem 4.1: as $\mathbb{C}$ is small and $\mathbf{D}$ is cocomplete, $\mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)$ exists; and as the Yoneda embedding is fully faithful, we know on top of that that the unit is thereof is an isomorphism. One can then set $\hat{F} := \mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F)$.

■

NB **Freeness of the construction**: Apart from the analogy with extension of continuous functions from dense subspaces in topology (since $\mathbf{y}_{\mathbb{C}}[\mathbb{C}]$ is "co-dense" in $\widehat{\mathbb{C}}$, in that every presheaf is a colimit of representables), one may wonder if this construction being *free* can be made precise in a categorical sense: i.e. is the cocompletion functor a left adjoint of a forgetful functor U?

It turns out that exhibiting such a left adjoint is more nettlesome than it may seem at first sight. Indeed,

- if the statement were purely 1-categorical (i.e. if the triangle commuted on the nose and the Yoneda extension $\hat{F}$ were actually unique (not just up to isomorphism)), then one could think that there is no problem whatsoever, for
  - $(\mathbb{C}, \mathbf{y}_{\mathbb{C}})$ would be an *initial arrow* from $\mathbb{C}$ to $U : \mathbf{Cocomp} \to \mathbf{Cat}$ (i.e. an initial object in $\Delta_{\mathbb{C}} \downarrow U$) for each category $\mathbb{C}$ (where **Cocomp** is the category of cocomplete categories with cocontinuous functors)
  - which is enough to have an adjunction (cf. Theorem B.1.6:

    $$(\hat{-}) \dashv U$$

  But we're not even in this situation here, as $\hat{F}$ is unique "up to isomorphism" (i.e. all the extensions of F that make the triangle commute up to natural isomorphism are isomorphic).
- One may then think that there are two possible workarounds:
  - to keep the commutativity and the uniqueness "on the nose", one may want to work with specified colimits, by considering the category $\mathbf{Cocomp}'$ of cocomplete categories equipped with a functor Colim that associates to each diagram (in a given cocomplete category) a particular colimit. The morphisms thereof would then be the cocontinuous functors that preserve the chosen colimits.
  - or one may settle for a 2-categorical statement: if **Cocomp** now denotes the 2-category of cocomplete categories (whose 1-arrows are cocontinuous functors and 2-arrows are natural transformations), and **Cat** the 2-category of (small) categories, one may venture that there is a 2-adjunction

    $$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "(\hat{-})"] \arrow[r, phantom, "\perp"] & \mathbf{Cocomp} \arrow[l, bend left, "U"] \end{tikzcd}$$

  But we're up a creek without a paddle anyway, as there is a size issue in any case: if $\mathbb{C}$ is small, then $\widehat{\mathbb{C}}$ is not small anymore in general (so the category **Cocomp** we're considering can't be the category of small cocomplete categories). As a result: $U(\widehat{\mathbb{C}})$ (which should be an object of **Cat**) is not a small category either!

Completeness: One may also wonder: ≪ where do the limits come from (as the presheaf category is also complete), given that we have only added the free colimits? ≫ Part of the reason why may be because of the following fact: every cocomplete category that has a small dense subcategory is complete (the dense subcategory here being the representables). There is a direct and elegant proof of this, by showing that the limit of a diagram $(\hat{P}_i)_i$ in $\widehat{\mathbb{C}}$ is nothing else than the colimit of the forgetful functor from the category of cones over $(\hat{P}_i)_i$ with summit an object of the dense subcategory.

Corollary B.1.15 — Every presheaf is a canonical colimit of representables For every $P \in \widehat{\mathbb{C}}$, where $\mathbb{C}$ is small:

$$P \cong \operatorname{colim}\left(\mathbf{y}_{\mathbb{C}} \downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}}\right)$$

Proof

Upon applying Theorem B.1.14 with $\mathbf{D} := \widehat{\mathbb{C}}, F := \mathbf{y}_{\mathbb{C}}$, it comes that

$$\operatorname*{Lan}_{\mathbf{y}_{\mathbb{C}}}(\mathbf{y}_{\mathbb{C}}) \cong \mathrm{Id}\widehat{\mathbb{C}}$$

by uniqueness (up to isomorphism) of $\hat{F}$.

Using the colimit expression of the Kan extension stemming from Theorem B.1.11 and taking the image at P yields the result. ■

B.2 Nerve construction: $\mathcal{E} \to \widehat{\mathbb{P}_+}$

In a 2008 article on the n-Category Café titled 'How I Learned to Love the Nerve Construction', Tom Leinster said:

> “The nerve construction is inherent in the theory of categories.”

And quite understandably: the nerve construction is an application of the Kan extension apparatus which unifies various parts of fields such as (higher) category theory, (higher) homotopy theory, algebraic topology, algebraic geometry, ... among others. To quote what Urs Schreiber wrote on the the corresponding nLab entry:

> “Pretty much every notion of category and higher category comes, or should come, with its canonical notion of simplicial nerve [...]”

And in our case, the nerve construction is precisely what will enable us to see event structures as presheaves over finite partial orders of events.

B.2.1 Nerve-Realisation paradigm

Let $F : \mathbb{C} \longrightarrow \mathbf{D}$ be a functor from a small category to locally small cocomplete one.

Proposition B.2.1

$$\underset{\mathbf{y}_{\mathbb{C}}}{\mathrm{Lan}}(F) \dashv \mathrm{N}_F \cong \underset{F}{\mathrm{Lan}}(\mathbf{y}_{\mathbb{C}})$$

Proof

As $\mathbf{D}$ is tensored and cocomplete, one can resort to the coend expression of the left Kan extension (cf. Theorem B.1.12):

- $\mathrm{Lan}_{\mathbf{y}_{\mathbb{C}}}(F) \dashv \mathrm{N}_F$:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}}\left(\underset{\mathbf{y}_{\mathbb{C}}}{\mathrm{Lan}}(F)(P), D\right) &\cong \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(c), P\right) \cdot Fc, D\right) \\
&\cong \mathrm{Hom}_{\mathbf{D}}\left(\int^{c} Pc \cdot Fc, D\right) \\
&\cong \int_{c} \mathrm{Hom}_{\mathbf{D}}\left(Pc \cdot Fc, D\right) \\
&\cong \int_{c} \mathrm{Hom}_{\mathbf{Set}}\left(Pc, \mathrm{Hom}_{\mathbf{D}}\left(Fc, D\right)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Set}}\left(P, \underbrace{\mathrm{Hom}_{\mathbf{D}}\left(F(-), D\right)}_{:= \mathrm{N}_F(D)}\right)
\end{aligned}$$

- $\mathrm{N}_F \cong \mathrm{Lan}_F(\mathbf{y}_{\mathbb{C}})$:

$$\begin{aligned}\operatorname*{Lan}_F(\mathbf{y}_{\mathbb{C}})(d) &\cong \int^c \mathrm{Hom}_{\mathbf{D}}\,(F(c), d) \cdot \mathbf{y}_{\mathbb{C}}(c)\\ &\cong \int^c \underbrace{\mathrm{Hom}_{\mathbf{D}}\,(F(c), d)}_{:=\mathrm{N}_F(d)(c)} \times \mathbf{y}_{\mathbb{C}}(c)\\ &\cong \int^c \mathbf{y}_{\mathbb{C}}(c) \times \mathrm{N}_F(d)(c)\\ &\cong \mathrm{N}_F(d) && \text{(co-Yoneda lemma)}\end{aligned}$$

■

B.2.2 Nerve of the inclusion of finite paths into event structures

Proposition B.2.2 Let $\mathbb{P}_+$ be the category of non-empty paths, and $\widehat{\mathbb{P}}'$ be the category of $\mathbb{P}$-presheaves A with $A\emptyset = \mathbb{1}$. We have an equivalence of categories $\widehat{\mathbb{P}}' \simeq \widehat{\mathbb{P}_+}$.

Proof
$\mathbb{P}_+$ is a full subcategory of $\mathbb{P}$, we denote by $\iota : \mathbb{P}_+ \hookrightarrow \mathbb{P}$ its fully faithful embedding into $\mathbb{P}$.

By restricting its domain, any presheaf of $\widehat{\mathbb{P}}'$ can be as presheaf over $\mathbb{P}_+$: we have a functor

$$\widehat{\mathbb{P}}' \xrightarrow{-\circ\iota} \mathbb{P}_+$$

Let's show that it is essentially surjective and fully faithful, which will be sufficient to get result, by Theorem B.1.1.

- $-\circ\iota$ **is essentially surjective**: indeed, any presheaf $A \in \widehat{\mathbb{P}_+}$ can be extended to a presheaf $A' \in \widehat{\mathbb{P}}'$ such that $A' \circ \iota = A$ by setting:
  - $A'(\emptyset) := \mathbb{1}$
  - $A'(\emptyset \xrightarrow{!_P} P) := A'(P) \xrightarrow{!_P} \mathbb{1}$ for all $P \in \mathbb{P}$
  - this extension clearly preserves the new identity $\mathrm{id}_\emptyset$, and we check that it still preserves composition:
    * for all $P \xrightarrow{f} Q$ in $\mathbb{P}$,

    $$A'(\underbrace{\emptyset \xrightarrow{!_P} P \xrightarrow{f} Q}_{=\emptyset \xrightarrow{!_Q} Q}) = A'(Q) \xrightarrow{!_Q} \mathbb{1} = A'(Q) \xrightarrow{A'(f)} A'(P) \xrightarrow{!_P} \mathbb{1} = A'(f) \ ; A'(!_P)$$

    * and there is no morphism whose codomain is $\emptyset$
- $-\circ\iota$ **is fully faithful**: for all $A, B \in \widehat{\mathbb{P}}'$, we clearly have:

$$\mathrm{Hom}_{\mathbb{P}_+}\,(A\iota, B\iota) \cong \mathrm{Hom}_{\widehat{\mathbb{P}}'}\,(A, B)$$

  as any natural transformation $\phi \in \mathrm{Hom}_{\widehat{\mathbb{P}}'}\,(A, B)$ cannot but be equal to $\mathrm{id}_\emptyset$ at $\emptyset$, since $A(\emptyset) = B(\emptyset) = \mathbb{1}$.

■