Ideal Distributors

Younesse Kaddar

Predoctoral Research Internship

école normale supérieure paris–saclay

Supervised by:

Marcelo Fiore

University of Cambridge

Computer Laboratory

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Overview

Research problem

In this report, we exhibit a lattice-theoretic model of linear logic (LL) comprised of what we call "ideal distributors". These ideal distributors stem from the consideration of Fiore and Joyal's cartesian closed bicategory of cartesian categories and cartesian distributors [FJ15]. Specifically, the differences are twofold (for reasons spelled out in section 1.3):

1. We restrict from cartesian categories to meet-semilattices.
2. We refine the presheaf construction (corresponding to downset completion in the posetal setting) to the ideal completion construction, which arises from an orthogonal subcategory construction.

The natural tensor product in the category of meet-semilattices is the one that classifies bihomomorphisms preserving cartesian structure. This adds a new layer of complexity to the development, because it differs from the traditional tensor of $\mathscr{V}$-categories used in presheaf constructions (underlying profunctorial models of LL, for example), which is merely pointwise.

From a computational point of view, this model is based on tokens of information forming a partial order that moreover has a meet-semilattice structure, allowing us to compute the maximum amount of information contained in any two pieces of information.

The question we tackle is whether we have a compact closed structure in this more sophisticated context, as it is known to be the case for $\mathscr{V}$-profunctors, where $\mathscr{V}$ is posetal. We prove that we do, and that we additionally have a full model of LL.

Contributions

- We show that the monad $\mathcal{I}\colon \boldsymbol{\mathcal{MDL}at} \to \boldsymbol{\mathcal{MDL}at}$ of ideals (non-empty directed downsets) on the category of bounded distributive lattices and meet-preserving maps is strong commutative for the tensor product classifying meet-preserving maps (that we restrict from the category of meet-semilattices to $\boldsymbol{\mathcal{MDL}at}$ thanks to a theorem of Fraser [Fra76, Theorem 2.6]).
- The previous fact enables us to endow the Kleisli category $\boldsymbol{\mathcal{IDL}at} := \mathcal{K}\ell(\mathcal{I})$ with a symmetric monoidal structure, and we go on to show that it is additionally *compact closed* (modeling the multiplicative fragment of classical LL in a degenerate way). This is the main technical result of this report, as the co/units of the compact closed structure are not the ones one might expect at first glance (generalising the compact closedness of $\mathbf{Rel}$, or quantale-enriched profunctors), which is due to the fact that the tensor product at hand is not the traditional tensor of $\mathscr{V}$-categories.
- Finally, we show that $\boldsymbol{\mathcal{IDL}at}$ constitutes a full model of classical LL, since it has biproducts (to model the additives) and free commutative monoids can be constructed explicitly in $\boldsymbol{\mathcal{IDL}at}$ (to model the exponentials). We additionally give a simplified expression of the dualisation operation arising from its compact closed structure.

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Prerequisites and notations: see appendix A.0.1.

1. Introduction

1.1 Profunctorial models of Linear Logic

Linear logic [Gir87] (LL) has strong links with denotational semantics and domain theory. In [See89], Seely showed that Barr's $*$-autonomous categories [Bar79] provide models of its classical multiplicative fragment. These are symmetric monoidal closed (SMC) categories $(\mathscr{C}, \otimes, \multimap)$ equipped with a dualising object $\perp \in \mathscr{C}$, in the sense that currying the canonical map $A \otimes (A \multimap \perp) \xrightarrow{\sigma} (A \multimap \perp) \otimes A \xrightarrow{\text{ev}} \perp$ yields an isomorphism $A \xrightarrow{\cong} ((A \multimap \perp) \multimap \perp)$ for all $A \in \mathscr{C}$; linear negation is then given by (internally) homing into $\perp$. A prototypical example of such $*$-autonomous structure is given by the category $\mathcal{SupLat}$ of **suplattices** (posets having all joins) [Bar91] and join-preserving maps – which are precisely left adjoints, by Freyd's adjoint theorem [FŠ90]. The tensor $\otimes$ represents maps preserving all joins componentwise (with the poset Σ of truth values as unit), the internal hom is the lattice of join-preserving maps with pointwise order, and the dualisation operation $(-)^* = (-)^\circ$ yields the opposite poset on objects and the left adjoint of the opposite map on morphisms.

Noteworthily, $\mathcal{SupLat}$ is simultaneously monadic over Set and over the category **Pos** of posets and monotone maps. Indeed, it can be seen as the the category of algebras

- $\mathbf{Set}^{\mathcal{P}}$ of the powerset monad $\mathcal{P}$, the Kleisli category $\mathbf{Set}_{\mathcal{P}}$ of which is equivalent to the category **Rel** of sets and relations, one of the simplest – yet fundamental – quantitative models of LL.
- $\mathbf{Pos}^{\mathcal{D}}$ of the monad of downward closed subsets (downsets) $\mathcal{D}$, the Kleisli category $\mathbf{Pos}_{\mathcal{D}}$ of which is equivalent to the category **RelPos** of posets and **relational profunctors**, that is, downward closed subsets $R \subseteq A^{\mathrm{op}} \times B$. This category plays an central role in Nygaard and Winskel's domain-theoretic approach to concurrency [NW; Win98].
  Incidently, it is also equivalent to the category of prime algebraic complete lattices and join-preserving maps (every prime algebraic complete lattice is isomorphic to the lattice of downsets of its complete primes) [Win09]. As such, **RelPos** is also sometimes denoted **ScottL** and constitutes another important model of LL, as it underlies the qualitative Scott model of prime algebraic lattices and Scott-continuous functions [Hut94; Win98].

Both of these full subcategories of $\mathcal{SupLat} \simeq \mathbf{Set}^{\mathcal{P}} \simeq \mathbf{Pos}^{\mathcal{D}}$ inherit its LL structure, which additionally becomes degenerate: they are **compact closed categories** (*i.e.* $*$-autonomous categories where the tensor $\otimes$ and its De Morgan dual, the 'par' $⅋$, coincide) of order-enriched profunctors. Recall that, given a Bénabou cosmos $(\mathscr{V}, \otimes)$ (a bicomplete SMC category), the bicategory $\mathscr{V}$–**Prof** of $\mathscr{V}$-enriched profunctor is given by $\mathscr{V}$-categories (0-cells), $\mathscr{V}$-functors $\mathscr{D}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{V}$ (1-cells $\mathscr{C} \to\!\!\!\mapsto \mathscr{D}$) and $\mathscr{V}$-enriched natural transformations (2-cells). Moreover, the "traditional" $\mathscr{V}$-categorical tensor product – defined on $\mathscr{V}$-categories $\mathscr{C}, \mathscr{D}$ by the category $\mathscr{C} \otimes \mathscr{D}$ (by abuse of notation) whose objects are pairs of objects and morphisms $(C_1, D_1) \to (C_2, D_2)$ are given by $\mathrm{Hom}_{\mathscr{C}}(C_1, C_2) \otimes \mathrm{Hom}_{\mathscr{D}}(D_1, D_2)$ – endows $\mathscr{V}$–**Prof** with a symmetric monoidal structure. For a general Bénabou cosmos, the bicategorical structure of $\mathscr{V}$–**Prof** stems from the composition $\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E}$ being defined by the coend formula $G \circ F := \int^{D \in \mathscr{D}} F(D, -) \otimes G(-, D)$, hence only up to isomorphism. As a result, in general, $\mathscr{V}$–**Prof** fails to be compact closed (in the traditional 1-categorical sense, not the bicategorical one [Sta16]) ≪ only because it fails to be an honest category with associative composition ≫, as Kelly and Laplaza put it in [KL80]. But when enriching over a *posetal* Bénabou cosmos[1] Q, also known as **quantale**, the bicategorical structure collapses and we get a genuine compact closed 1-category, which is equivalent to the Kleisli category of the free Q-enriched cocompletion monad.

Now, **RelPos** is a special case of $\mathscr{V}$–**Prof** where $\mathscr{V} := \Sigma = \perp \to \top$ (the interval category), and the linear logic tensor is the corresponding traditional tensor of Σ-categories. Restricting Σ–**Prof** to discrete small categories (*i.e.* sets) leads to the subcategory **Rel**, which inherits the LL structure. Various categorification of these LL models have been studied in the Set-enriched setting.

[1] equivalently: a monoid in $\mathcal{SupLat}$

Example 1.1

- `Categorification of the relational model:` Fiore, Gambino, Hyland and Winskel's generalised species of structures [Fio+08] is a bicategorical LL model of **Set**-enriched profunctors generalising Joyal's combinatorial species of structures [Joy81], where the exponential is given by free symmetric monoidal completion.
- `Categorification of the Scott model:`
  - In [CW05], Cattani and Winskel endowed the bicategory of **Set**-enriched profunctors with the free finite colimit completion exponential modality, yielding a model of linear logic drawing upon the Scott model, where directed joins are categorified as filtered colimits, and Scott-continuous functions as finitary functors.
  - Recently, Galal [Gal20] offered another take on generalising the Scott model, with the free finite coproduct completion exponential modality: directed joins are categorified as sifted colimits, and Scott-continuity by strongly finitary functors. Her $\mathcal{C}$-species are the subcartesian closed bicategory of Fiore and Joyal's cartesian closed bicategory of cartesian distributors restricted to the free objects [FJ15].

Enriching profunctors over an arbitrary quantale has also proved fruitful in the context of resource theory, as shown by Marsden and Zwart [MZ18], and in Censi's theory of co-design [Cen16; FS19].

1.2 Ideal completion

From a computational point of view, in **Rel**, tokens (elements) have no structure. Now, suppose that we add extra structure given by meets (we can compute the maximum amount of information contained in two pieces of information), in a similar fashion to Pratt's state spaces [Pra], and we require that this structure be preserved by morphisms. Instead of considering presheaves (as in all the previous examples), corresponding to *all* monotone maps $A^{\mathrm{op}} \longrightarrow \Sigma$ in the Σ-enriched setting, this new requirement singles out *finite meet preserving* maps $A^{\mathrm{op}} \longrightarrow \Sigma$. And provided that A has finite meets and joins, these are in one-to-one correspondence with the set $\mathcal{I}(A)$ of order-theoretic **ideals** (non-empty directed downsets) in A.

NB We draw attention to the fact that there is a clash of terminology when it comes to the very definition of an 'ideal' in a preorder. In concurrency theory à la Winskel (and various other sources, such as Pratt's event spaces paper [Pra]), downsets are called 'ideals', so that Σ-enriched profunctors can be seen as monotone maps into a poset of ideals. In the present report however, 'ideal' will always be meant in the order-theoretic sense, *i.e.* non-empty *directed* downset.

Ideals play a key role in domain theory [AGM92; Gie03], since ideal completion is a universal way to generate domains.

Example 1.2 In the early days of domain theory, Scott used the cartesian closed category of continuous lattices (complete lattices where directed joins commute with arbitrary meets) and Scott-continuous functions to provide models of the untyped λ-calculus. The free continuous lattice on a poset can be obtained by free meet completion (which amounts to taking arbitrary upsets with reverse inclusion) followed by ideal completion.

From a domain-theoretic standpoint, ideal completion amounts to freely adding directed joins, which can be interpreted as freely adding elements amalgamating the information of each directed subset, seen as a set of partial computation results (or pieces of information) that are pairwise compatible.

As such, every *Scott domain* (bounded-complete algebraic pointed dcpo) is the ideal completion of its compact elements [Sco82], and the category of Scott domains and linear maps forms a model of intuitionistic LL (the exponential being obtained as an ideal completion). The resulting co-Kleisli category is the traditional category of Scott domains and Scott continuous maps, which is of paramount importance in denotational semantics.

In the same way as Σ-enriched profunctors (also called distributors) correspond to monotone maps $\mathscr{C} \longrightarrow [\mathscr{D}^{\mathrm{op}}, \Sigma]$, meet-preserving maps $\mathscr{C} \longrightarrow \mathcal{I}(\mathscr{D})$ correspond to what we will refer to as **ideal distributors**. Note that the tensor we are then considering is *not* the traditional tensor of $\mathscr{V}$-categories (for $\mathscr{V} := \Sigma$), but the tensor representing bihomomorphisms, *i.e.* monotone maps that preserve finite meets componentwise. As a consequence, the classical results about Q-enriched profunctors (for Q a quantale) do not easily carry over, especially compact closedness.

1.3 Orthogonal construction

In this expository section, we motivate the technical assumptions that will be made in the subsequent sections. For more details, we refer the reader to [Bor94; Bor08; Kel], and [Fio96, Subsection 11.2].

Let $\mathbb{C}$ be a small category, and consider its embedding $\mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ into its free cocompletion: the category obtained from $\widehat{\mathbb{C}}$ by freely adding all small colimits. In general, this embedding does not preserve colimits that exist in $\mathbb{C}$. Indeed, let $A := \operatorname{colim}_i D_i \in \mathbb{C}$ be the colimit of a diagram $D\colon I \longrightarrow \mathbb{C}$. If we regard $\mathbb{C}$ as a subcategory of $\widehat{\mathbb{C}}$, A is sent to itself by the embedding in $\widehat{\mathbb{C}}$, whereas the colimit of $D\colon I \longrightarrow \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ is the *formal colimit* (freely added by free cocompletion) "colim_i"$D_i \in \widehat{\mathbb{C}}$. And while $\mathrm{Hom}_{\widehat{\mathbb{C}}}(B, \text{“}\operatorname{colim}_i\text{”} D_i) \cong \operatorname{colim}_i \mathrm{Hom}_{\mathbb{C}}(B, D_i)$ for every $B \in \mathbb{C}$ by definition, this does not necessarily hold for the original colimit $A = \operatorname{colim}_i D_i$. Otherwise, we would have, for $B := A \in \mathbb{C}$: $\{*\} \cong \mathrm{Hom}_{\widehat{\mathbb{C}}}(A, \operatorname{colim}_i D_i) \cong \operatorname{colim}_i \mathrm{Hom}_{\mathbb{C}}(\operatorname{colim}_j D_j, D_i) \cong \operatorname{colim}_i \lim_j \mathrm{Hom}_{\mathbb{C}}(D_j, D_i)$, which is clearly not true in general.

Sometimes, this non-preservation is problematic at several levels, since the non-trivial information carried by existing colimits in $\mathbb{C}$ may be lost when trading them for "syntactical" colimits. A typical example thereof is space gluing, in an appropriate category of spaces $\mathbb{C}$: Grothendieck's approach resorting to sheaves precisely addresses[2] this issue [Bor08; Dug; MM92]. As a matter of fact, preserving a certain class of existing colimits in $\mathbb{C}$ can be seen as a refinement of free cocompletion that respects the structure of $\mathbb{C}$ to some extent, rather than blithely adding to $\mathbb{C}$ all formal colimits, thereby "overwriting" the said structure in $\widehat{\mathbb{C}}$.

Let Φ be a class of (small) cocones in $\mathbb{C}$. Suppose that these cocones are colimiting in $\mathbb{C}$ (we will say that $\mathbb{C}$ has Φ-*colimits*): it is well known that the Yoneda embedding $\mathbf{y}_{\mathbb{C}}(-) := \mathrm{Hom}_{\mathbb{C}}(=, -)\colon \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ from $\mathbb{C}$ to the category $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on $\mathbb{C}$ exhibits $\widehat{\mathbb{C}}$ as the free cocompletion of $\mathbb{C}$. By continuity of the Hom functor, $\mathbf{y}_{\mathbb{C}}$ preserves small limits, but does not preserve colimits in general, as argued before. A natural question arises then: how to find a free full subcategory $\widetilde{\mathbb{C}} \hookrightarrow \widehat{\mathbb{C}}$ such that the corestriction of the Yoneda embedding to $\widetilde{\mathbb{C}}$ preserves Φ-colimits?

$$\begin{tikzcd} \mathbb{C} \arrow[rr, hook, "\mathbf{y}_{\mathbb{C}}"] \arrow[dr, hook, "\substack{\mathbf{y}_{\mathbb{C}} \\ \text{preserving} \\ \Phi\text{-colimits}}"'] & & \widehat{\mathbb{C}} \\ & \widetilde{\mathbb{C}} \arrow[ur, hook] & \end{tikzcd}$$

The answer, that may seem tautological at first glance, is to require that $\widetilde{\mathbb{C}}$ be the full subcategory $\mathcal{O}(\widehat{\mathbb{C}}, \Phi)$ of all those presheaves that "believe" that all the Φ-cocones are indeed colimiting ones in $\widehat{\mathbb{C}}$. Formally, this is encapsulated in the definition of **orthogonal** objects:

Definition 1.1 An object $B \in \mathscr{C}$ in a category $\mathscr{C}$ is **orthogonal** to a cocone $\gamma\colon D \overset{\cdot}{\longrightarrow} C$ in $\mathscr{C}$, denoted $\gamma \perp B$, iff every cocone $D \overset{\cdot}{\longrightarrow} B$ factors uniquely through γ. If Φ is a class of small cocones in $\mathscr{C}$, let $\mathcal{O}(\mathscr{C}, \Phi) \subseteq \mathscr{C}$ denote the full subcategory spanned by the objects orthogonal to every cocone in Φ.

Example 1.3 This notion directly extends that of orthogonality to maps in $\mathscr{C}$, a typical example thereof being given by sheaves, as alluded before. The category $\mathrm{Sh}(\mathbb{C}) \hookrightarrow \widehat{\mathbb{C}}$ of sheaves on a small site $(\mathbb{C}, J)$ is the full subcategory of presheaves that are orthogonal to every covering sieve $S \longrightarrow \mathbf{y}_{\mathbb{C}}(U)$ [nLab]. Indeed, for every $F \in \widehat{\mathbb{C}}$, $\mathrm{Hom}_{\widehat{\mathbb{C}}}(S, F)$ is the equaliser of $\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$, so that the fact that every map $S \longrightarrow F$ factors through $\mathbf{y}_{\mathbb{C}}(U) \longrightarrow F$ (corresponding to an element of $F(U)$ by the Yoneda

[2] in the sheaf-theoretic approach, this deficiency is what underlies the specificities of sieves.

lemma) is tantamount to the locality and gluing sheaf properties.

If all cocones in Φ are colimiting, $\mathcal{O}(\widehat{\mathbb{C}}, \Phi)$ turns out to be the free cocompletion of $\mathbb{C}$ that respects Φ-colimits, in the sense that:

Theorem 1.1 — Free cocompletion respecting Φ-colimits [Fio96; Kel]. If Φ is a class of colimiting (small) cocones in a small category $\mathbb{C}$,

- $\mathcal{O}(\widehat{\mathbb{C}}, \Phi)$ is cocomplete and the Yoneda embedding corestricts to a Φ-cocontinuous (*i.e.* Φ-colimit preserving) embedding $\mathbb{C} \hookrightarrow \mathcal{O}(\widehat{\mathbb{C}}, \Phi)$
- such that every Φ-cocontinuous functor $\mathbb{C} \longrightarrow \mathscr{D}$ into a cocomplete category $\mathscr{D}$ factors uniquely through $\mathbb{C} \hookrightarrow \mathcal{O}(\widehat{\mathbb{C}}, \Phi)$ via a Φ-cocontinuous functor $\mathcal{O}(\widehat{\mathbb{C}}, \Phi) \longrightarrow \mathscr{D}$.

As it happens, the functors $\mathcal{O}(\widehat{\mathbb{C}}, \Phi) \longrightarrow \mathscr{D}$ are precisely those which have a left-adjoint, exhibiting $\mathcal{O}(\widehat{\mathbb{C}}, \Phi)$ as a reflective subcategory of $\widehat{\mathbb{C}}$. And the orthogonal subcategory problem (asking when is $\mathcal{O}(\widehat{\mathbb{C}}, \Phi)$ a reflective subcategory) is known to be tighly linked to the continuous functor problem (asking when is the category of functor preserving limits of certain shapes reflective) [FK72], which shows through the following proposition [Fio96, Proposition 11.4]:

Proposition 1.1 If Φ is a class of (small) cocones in a small category $\mathbb{C}$ and $\widehat{\Phi} := \{\mathbf{y}_{\mathbb{C}}\gamma \mid \gamma \in \Phi\}$, $\mathcal{O}(\widehat{\mathbb{C}}, \widehat{\Phi}) \hookrightarrow \widehat{\mathbb{C}}$ is the full subcategory spanned by the presheaves P such that for every $\gamma \in \Phi$, $P\gamma$ is limiting in **Set**.

The special case that will be of particular interest to us is when the category $\mathbb{C} \in \mathscr{K}$ has finite coproducts and is an object of an ambient bicategory $\mathscr{K}$ acting as a higher-categorical model of LL or domains. Let Φ_+ be the class of finite coproduct cocones. In this case, by Proposition 1.1, $\widetilde{\mathbb{C}} := \mathcal{O}(\widehat{\mathbb{C}}, \widehat{\Phi_+})$ is the category $\mathrm{FProd}\,(\mathbb{C}^{\mathrm{op}}, \mathbf{Set})$ of finite product preserving presheaves, leading to the following situation:

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, hook, "\mathbf{y}_{\mathbb{C}}"] \arrow[dr, hook, "\substack{\mathbf{y}_{\mathbb{C}}\\ \text{preserving}\\ \text{finite}\\ +}"'] & & \widehat{\mathbb{C}} \arrow[dl, dotted, bend right] \arrow[dl, phantom, "\perp"] \\
& \mathrm{FProd}\,(\mathbb{C}^{\mathrm{op}}, \mathbf{Set}) \arrow[ur, hook, bend right] &
\end{tikzcd}$$

Now, by Day's reflection theorem [Day72], every reflective subcategory $\widetilde{\mathscr{C}}$ of a symmetric monoidal closed category $(\mathscr{C}, \otimes)$ inherits its monoidal closed structure provided that $\widetilde{\mathscr{C}}$ is an **exponential ideal**, *i.e.* for every $A \in \widetilde{\mathscr{C}}$ and $C \in \mathscr{C}$, $[C, A] \in \widetilde{\mathscr{C}}$. And in our particular case, where $(\mathscr{C} := \widehat{\mathbb{C}}, \times)$ is cartesian closed, this results in a full sub-cartesian-closed-category of $\widehat{\mathbb{C}}$.

To ensure that $\widetilde{\mathbb{C}}$ is an exponential ideal, a sufficient condition is that $\mathbb{C}$ be a distributive category (proof in appendix):

Proposition — B.1 Suppose $\mathbb{C}$ is a **distributive category**, *i.e.* has finite products $\times$ and coproducts $+$ such that for every $A, B, C \in \mathbb{C}$, the canonical morphism $A \times B + A \times C \longrightarrow A \times (B + C)$ is invertible. Then $\mathrm{FProd}\,(\mathbb{C}^{\mathrm{op}}, \mathbf{Set})$ is an exponential ideal of $\widehat{\mathbb{C}}$.

Distributive categories are not self dual, implying that the ambient category $\mathscr{K}$ is not closed under the dualisation operation $(-)^{\mathrm{op}}$, as desired, if it were to be a category of distributive categories. This leads us to restrict from categories to preorders (and even posets, for convenience), assuming that our categories are Σ-enriched rather than **Set**-enriched, where $\Sigma := \bot \longrightarrow \top$ is the interval category.

In this setting, $\mathbb{C}$ will be a bounded distributive lattice, $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \Sigma]$ the lattice of downsets of $\mathbb{C}$, and $\mathrm{FProd}\,(\mathbb{C}^{\mathrm{op}}, \Sigma)$ the lattice of (finite) meet preserving maps from $\mathbb{C}$ to Σ, which are in one-to-one correspondence with **order-theoretic ideals** – non-empty down-closed and directed subsets – by considering the preimage of $\top \in \Sigma$. This ideal completion $\mathcal{I}(\mathbb{C})$ of $\mathbb{C}$ amounts to freely adding all (small) directed colimits [JJ82], and hence all filtered colimits (also known as **Ind-completion**) [AR94].

2. Ideal Distributors

2.1 Categories of posets

We introduce a handful of categories that we will use throughout this report.

Definition 2.1

- $\mathcal{MSLat}$ is the category of **bounded meet semilattices** (posets with finite meets $\wedge$ and a greatest element $\top$) and monotone functions that preserve finite meets,
- $\mathcal{MDLat} \hookrightarrow\!\!\!\!\rightarrow \mathcal{MSLat}$ is its full subcategory consisting of the **bounded distributive lattices** (posets with finite meets and finite joins that distribute over each other, having a least (resp. greatest) element $\perp$ (resp. $\top$)) with meet-preserving maps,
- $\mathcal{DLat} \hookrightarrow \mathcal{MSLat}$ is the subcategory of bounded distributive lattices and lattice (meet- and join-preserving) morphisms.
- $\mathcal{Frm} \hookrightarrow \mathcal{Frm}^{\uparrow}$ are, respectively, the category of **frames** (posets with finite meets and all joins with the former distributing over the latter) and monotone functions that preserve finite meets and all (resp. directed) joins.

2.2 Monad of order-theoretic ideals

The previously mentioned *ideal monad* that will be central in our investigation arises from the free-forgetful adjunction between $\mathcal{Frm}^{\uparrow}$ and $\mathcal{MDLat}$:

Proposition 2.1 The forgetful functor $\mathcal{Frm}^{\uparrow} \longrightarrow \mathcal{MDLat}$ has a left adjoint.

For a distributive lattice D, the free construction $D \longrightarrow \mathcal{I}(D)$ can be explicitly described by taking $\mathcal{I}(D)$ to be the poset of ideals (non-empty directed down-sets) of D ordered by inclusion with intersections as meets and ideal generated by unions as joins (the bottom element being the intersection of all the ideals). For every morphism $f\colon D \longrightarrow D'$,

$$\mathcal{I}(f) \;:=\; \begin{cases} \mathcal{I}(D) & \longrightarrow \mathcal{I}(D') \\ \delta & \longmapsto \{d' \in D' \mid \exists\, d \in \delta;\ d' \leq f(d)\} \end{cases}$$

Poscomposition by the forgetful functor yields a monad $\mathcal{I}\colon \mathcal{MDLat} \longrightarrow \mathcal{MDLat}$ that we will call the **ideal monad**, whose multiplication μ and unit η are defined as follows, for all $D \in \mathcal{MDLat}$ (see Proposition C.1):

$$\eta_D \;:=\; \begin{cases} D & \longrightarrow \mathcal{I}(D) \\ d & \longmapsto {\downarrow} d \end{cases} \qquad \mu_D \;:=\; \begin{cases} \mathcal{I}(\mathcal{I}(D)) & \longrightarrow \mathcal{I}(D) \\ \Phi & \longmapsto \{d \in D \mid {\downarrow} d \in \Phi\} \end{cases}$$

Definition 2.2 Let $\mathcal{IDLat}$ be the Kleisli category $\mathcal{K}\ell(\mathcal{I})$ of the ideal monad on $\mathcal{MDLat}$.

Notation 2.1. *Let* Σ *denote the distributive lattice* $(\perp \leq \top)$*, sometimes called the* Sierpiński space.

Proposition — C.2 The category $\mathcal{IDLat}$ can be equivalently described as that with bounded distributive lattices as objects and morphisms given by distributors $f\colon X \relbar\!\!\mapsto Y$: that is, monotone functions $f\colon Y^{\circ} \times X \longrightarrow \Sigma$ such that, for all $x \in X$ and $y \in Y$, $f(-, x) : Y^{\circ} \longrightarrow \Sigma$ and $f(y, -) : X \longrightarrow \Sigma$ preserve finite meets, with identities given by $\mathrm{id}(x', x) = [\, x' \leq x\,]$ and composition $f \circ g\colon X \relbar\!\!\mapsto Z$ of

$g : X \rightarrow\!\!\!\!\!\!\!\!\;\;\shortmid\;\; Y$ and $f : Y \rightarrow\!\!\!\!\!\!\!\!\;\;\shortmid\;\; Z$ given by

$$(f \circ g)(z, x) = \bigvee_{Z_0 \subseteq_{\text{fin}} Z} \left[z \leq \bigvee Z_0\right] \wedge \left[\, \bigwedge_{z_0 \in Z_0} \bigvee_{y \in Y} f(z_0, y) \wedge g(y, x) \,\right] . \tag{2.1}$$

Intuitively, *eq.* (2.1) says that $z \in Z$ is related to $x \in X$ via $f \circ g$ when there exists a finite cover of z such that each element in there is related via f to some $y \in Y$ that is in turn related to x via g.

2.3 Symmetric monoidal structure

We will show that the symmetric monoidal structure of $\mathcal{IDLat}$ is inherited – in a canonical way – from that of $\mathcal{MDLat}$, itself inherited from that of $\mathcal{MSLat}$ (thanks to a key result of Fraser [Fra76, Theorem 2.6]).

Proposition 2.2 The category $\mathcal{MDLat}$ is symmetric monoidal.

Proof
The tensor unit is Σ and the tensor product is given by the construction of the universal bihomomorphism $X \times Y \longrightarrow X \otimes Y$ in $\mathcal{MSLat}$, which happens to yield a distributive lattice whenever X and Y are distributive; see [Fra76, Theorem 2.6]. $X \otimes Y$ is the meet-semilattice generated by the elements $x \otimes y$ ($x \in X$, $y \in Y$), subject to the relations, for all $x, x_1, x_2 \in X,\ y, y_1, y_2 \in Y$:

$$\top \otimes y = \top \qquad\qquad (x_1 \wedge x_2) \otimes y = (x_1 \otimes y) \wedge (x_2 \otimes y)$$

and

$$x \otimes \top = \top \qquad\qquad x \otimes (y_1 \wedge y_2) = (x \otimes y_1) \wedge (x \otimes y_2)$$

Every element of $X \otimes Y$ is of the form $\bigwedge_{1 \leq i \leq n} x_i \otimes y_i$ for some $x_i \in X$, $y_i \in Y$, $i \in [\![1, n]\!]$. We allow the tensor $\otimes$ to bind more tightly than meets $\wedge$ and joins $\vee$. ■

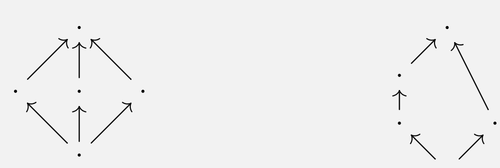
NB Note also from [Fra76] that the distributive lattice $X \otimes Y$ is isomorphic to the free product – *i.e.* the coproduct in our case – of X and Y in the subcategory of $\mathcal{MDLat}$ consisting of the bottom-preserving homomorphisms. Henceforth, when referring to Fraser's results [Fra76], we will implicitely resort to the dual versions, as he is working in the category of join-semilattices.

At first glance, one might think that $\mathcal{MDLat}$ is bicomplete, because distributive lattices are algebras of a Lawvere theory, which implies that their category is bicomplete (see [ARV11; nLaa]). But this does not hold in our case, as morphisms need not preserve joins in $\mathcal{MDLat}$ (so we are *not* considering the category of algebras of the distributive lattice monad). And this fact is precisely what underlies the two following counter-examples:

Proposition 2.3 $\mathcal{MDLat}$ is neither complete nor cocomplete.

Proof
We will show that it does not have equalisers nor coequalisers. The two archetypal examples of non-distributive lattices[1] will be called the **diamond lattice** and the **pentagon lattice**:

Example 2.1 — Non-distributive lattices: Diamond lattice and Pentagon lattice

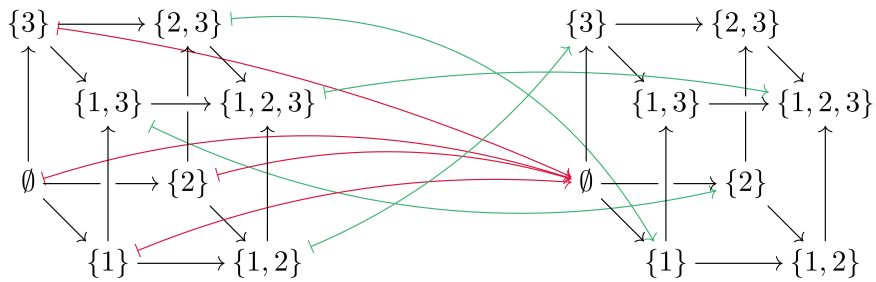


[1] in that a lattice is non-distributive iff one of its sublattices is isomorphic to one of them

To show that $\boldsymbol{\mathcal{MDLat}}$ does not have co/equalisers, we will exhibit a co/equaliser of distributive lattices which is not distributive.

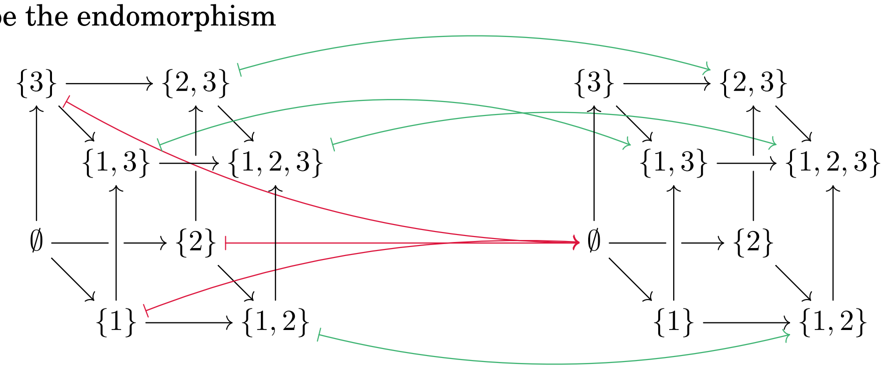
- $\boldsymbol{\mathcal{MDLat}}$ `does not have equalisers`: In [Barb], Michael Barr gives a counter-example, that we dualise here. Let 2^3 be the powerset of a three-element set (with set intersection and union as meet and join), and $f, g\colon 2^3 \longrightarrow 2^3$ two endomorphisms, where g maps the top element $\{1,2,3\}$ to $\{1,2,3\}$ and every other element to the bottom element $\emptyset$; and f is defined as:

$f\colon$



  It is not hard to check that the equaliser of f and g in $\boldsymbol{\mathcal{MSLat}}$ is the diamond lattice.
- $\boldsymbol{\mathcal{MDLat}}$ `does not have coequalisers`: We can come up with an analogous counter-example. Let $h\colon 2^3 \longrightarrow 2^3$ be the endomorphism

$h\colon$



  Likewise, it is routine to check that the coequaliser of h and id_{2^3} in $\boldsymbol{\mathcal{MSLat}}$ is the diamond lattice.

NB In the previous counter-examples, neither f nor h preserve joins ($f(\{1\} \cup \{3\}) = \{2\} \neq \emptyset = f(\{1\}) \cup f(\{3\})$ and $h(\{1\} \cup \{3\}) = \{1,3\} \neq \emptyset = h(\{1\}) \cup h(\{3\})$).

■

2.4 The ideal monad is strong commutative

We now recall the definition of a strong commutative monad and will go on to show that $\boldsymbol{\mathcal{IDLat}}$ is such a monad. The enticing result one would ideally want is the following:

Lemma 2.1 — [Hyl+06] Example 3.12. If T is a commutative monad on a cocomplete symmetric monoidal closed category $\mathscr{C}$, then the free commutative monoids monad on $\mathscr{C}$ extends to $\mathcal{K}\ell(T)$.

Unfortunately, in our case, $\boldsymbol{\mathcal{MDLat}}$ may not even be closed for the tensor product (see Barr's MathOverflow question [Bara]). But showing that $\mathcal{I}$ is strong commutative will still prove useful to endow $\mathcal{K}\ell(\mathcal{I})$ with a symmetric monoidal structure.

Definition 2.3 — A strong monad (T, η, μ) over a monoidal category $(\mathscr{C}, \otimes, I)$ is a monad equipped with a natural transformation $t_{A,B}\colon A \otimes TB \longrightarrow T(A \otimes B)$ called **strength** such that the following diagrams commute for all $A, B, C \in \mathscr{C}$:

$$\begin{tikzcd} I \otimes TA \arrow[r, "t_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & T(I \otimes A) \arrow[d, "T(\lambda_A)"] \\ & TA \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B) \otimes TC \arrow[rr, "t_{A\otimes B,C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \otimes B) \otimes C) \arrow[d, "T(\alpha_{A,B,C})"] \\ A \otimes (B \otimes TC) \arrow[r, "1 \otimes t_{B,C}"'] & A \otimes T(B \otimes C) \arrow[r, "t_{A,B\otimes C}"'] & T(A \otimes (B \otimes C)) \end{tikzcd}$$

$$\begin{tikzcd} A \otimes B \arrow[r, "\mathrm{id}\otimes\eta_B"] \arrow[dr, "\eta_{A\otimes B}"'] & A \otimes TB \arrow[d, "t_{A,B}"] \\ & T(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} A \otimes T^2 B \arrow[r, "t_{A,TB}"] \arrow[d, "1 \otimes \mu_B"'] & T(A \otimes TB) \arrow[r, "T(t_{A,B})"] & T^2(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\ A \otimes TB \arrow[rr, "t_{A,B}"'] & & T(A \otimes B) \end{tikzcd}$$

If $(\mathscr{C}, \otimes, I, \sigma)$ is symmetric, a strong monad (T, η, μ) is said to be **commutative** if

$$\begin{tikzcd} TA \otimes TB \arrow[r, "t_{TA,B}"] \arrow[d, "t'_{A,TB}"'] & T(TA \otimes B) \arrow[r, "T(t'_{A,B})"] & T^2(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\ T(A \otimes TB) \arrow[r, "T(t_{A,B})"'] & T^2(A \otimes B) \arrow[r, "\mu_{A\otimes B}"'] & T(A \otimes B) \end{tikzcd}$$

commutes for all $A, B \in \mathscr{C}$, where $t'_{A,B} := TA \otimes B \xrightarrow{\sigma_{A,B}} B \otimes TA \xrightarrow{t_{B,A}} T(B \otimes A) \xrightarrow{T(\sigma_{B,A})} T(A \otimes B)$ is called a **costrength**. The costrength satisfies similar diagrams to the strength ones (see [nLac]).

Proposition — C.3 $\mathcal{I}$ is a strong monad.

Proof
The strength $t_{A,B} \colon A \otimes \mathcal{I}(B) \longrightarrow \mathcal{I}(A \otimes B)$, given by universal property of the tensor product for the bihomomorphism

$$\tilde{t}_{A,B} \colon \begin{cases} A \times \mathcal{I}(B) & \longrightarrow \mathcal{I}(A \otimes B) \\ a,\, \emptyset \neq \beta \subseteq B & \longmapsto \downarrow \{a \otimes b\}_{b \in \beta} \end{cases}$$

makes $\mathcal{I}$ strong. ■

Lemma — C.1. $\mathcal{I}$ is a commutative monad.

Proof
Showing that the costrength

$$t'_{A,B} := \mathcal{I}A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes \mathcal{I}A \xrightarrow{t_{B,A}} \mathcal{I}(B \otimes A) \xrightarrow{\mathcal{I}(\sigma_{B,A})} \mathcal{I}(A \otimes B)$$

can be expressed as

$$t'_{A,B} \colon \begin{cases} \mathcal{I}(A) \otimes B & \longrightarrow \mathcal{I}(A \otimes B) \\ \bigwedge_i \alpha_i \otimes b_i & \longmapsto \bigcap_i \downarrow \{a \otimes b_i\}_{a \in \alpha_i} \end{cases}$$

enables us to prove that the relevant square commutes without much effort (see Lemma C.1). ■

Corollary 2.2 By Lemma C.1, $\mathcal{IDLat} := \mathcal{K}\ell(\mathcal{I})$ inherits a symmetric monoidal structure.

Proof
This results from the fact that $\mathcal{MDLat}$ is symmetric monoidal and $\mathcal{I}$ is commutative, see [PR95, Corollary 4.3.] or [Gui80; Sea]. The tensor $\otimes$ is given

- on objects, by the underlying tensor in $\mathcal{MDLat}$
- on morphisms $f\colon A \multimap B = A \longrightarrow \mathcal{I}B,\ g\colon A' \multimap B' = A' \longrightarrow \mathcal{I}B'$, by

$$f \otimes g = A \otimes A' \xrightarrow{f\otimes g} \mathcal{I}B \otimes \mathcal{I}B' \xrightarrow{\mu_{B\otimes B'}\,\mathcal{I}(t'_{B,B'})\,t_{\mathcal{I}B,B'} = \mu_{B\otimes B'}\,\mathcal{I}(t_{B,B'})\,t'_{B,\mathcal{I}B'}} \mathcal{I}(B \otimes B')$$

To show that $\otimes$ is a bifunctor, Power and Robinson resort to the commutativity of the monad. The associator and left/right unitors are given by:

$$\begin{array}{rl}
& (A\otimes B)\otimes C \xrightarrow{1\otimes\eta_C} (A\otimes B)\otimes \mathcal{I}C \xrightarrow{\mathcal{I}(\alpha_{A,B,C})\,t_{A\otimes B,C} = t_{A,B\otimes C}\,(1\otimes t_{B,C})\,\alpha_{A,B,\mathcal{I}C}} \mathcal{I}(A\otimes(B\otimes C)) \\
= & (A\otimes B)\otimes C \xrightarrow{\eta_{(A\otimes B)\otimes C}} \mathcal{I}((A\otimes B)\otimes C) \xrightarrow{\mathcal{I}(\alpha_{A,B,C})} \mathcal{I}(A\otimes(B\otimes C)) \\
= & (A\otimes B)\otimes C \xrightarrow{\alpha_{A,B,C}} A\otimes(B\otimes C) \xrightarrow{\eta_{A\otimes(B\otimes C)}} \mathcal{I}(A\otimes(B\otimes C)) \\
& \Sigma\otimes A \xrightarrow{1\otimes\eta_A} \Sigma\otimes\mathcal{I}A \xrightarrow{\lambda_{\mathcal{I}A}} \mathcal{I}A \;=\; \Sigma\otimes A \xrightarrow{\eta_{\Sigma\otimes A}} \mathcal{I}(\Sigma\otimes A) \xrightarrow{\mathcal{I}(\lambda_A)} \mathcal{I}A \;=\; \Sigma\otimes A \xrightarrow{\lambda_A} A \xrightarrow{\eta_A} \mathcal{I}A \\
& A\otimes\Sigma \xrightarrow{\eta_A\otimes 1} \mathcal{I}A\otimes\Sigma \xrightarrow{\rho_{\mathcal{I}A}} \mathcal{I}A \;=\; A\otimes\Sigma \xrightarrow{\eta_{A\otimes\Sigma}} \mathcal{I}(A\otimes\Sigma) \xrightarrow{\mathcal{I}(\rho_A)} \mathcal{I}A \;=\; A\otimes\Sigma \xrightarrow{\rho_A} A \xrightarrow{\eta_A} \mathcal{I}A
\end{array}$$

Their inverses are:

$$\begin{array}{rl}
& A\otimes(B\otimes C) \xrightarrow{\eta_C\otimes 1} \mathcal{I}A\otimes(B\otimes C) \xrightarrow{\mathcal{I}(\alpha^{-1}_{A,B,C})\,t'_{A,B\otimes C} = t'_{A\otimes B,C}\,(t'_{A,B}\otimes 1)\,\alpha^{-1}_{\mathcal{I}A,B,C}} \mathcal{I}((A\otimes B)\otimes C) \\
= & A\otimes(B\otimes C) \xrightarrow{\eta_{A\otimes(B\otimes C)}} \mathcal{I}(A\otimes(B\otimes C)) \xrightarrow{\mathcal{I}(\alpha^{-1}_{A,B,C})} \mathcal{I}((A\otimes B)\otimes C) \\
= & A\otimes(B\otimes C) \xrightarrow{\alpha^{-1}_{A,B,C}} (A\otimes B)\otimes C \xrightarrow{\eta_{(A\otimes B)\otimes C}} \mathcal{I}((A\otimes B)\otimes C) \\
& A \xrightarrow{\eta_A} \mathcal{I}(A) \xrightarrow{\mathcal{I}\lambda^{-1}_A} \mathcal{I}(\Sigma\otimes A) = A \xrightarrow{\lambda^{-1}_A} \Sigma\otimes A \xrightarrow{\eta_{\Sigma\otimes A}} \mathcal{I}(\Sigma\otimes A) \\
& A \xrightarrow{\eta_A} \mathcal{I}(A) \xrightarrow{\mathcal{I}(\rho^{-1}_A)} \mathcal{I}(A\otimes\Sigma) = A \xrightarrow{\rho^{-1}_A} A\otimes\Sigma \xrightarrow{\eta_{A\otimes\Sigma}} \mathcal{I}(A\otimes\Sigma)
\end{array}$$

Indeed:

$$\overbrace{1\otimes\eta\,;\,t}^{=\eta}\,;\,\mathcal{I}\alpha\,;\,\overbrace{\mathcal{I}(\eta\otimes 1)\,;\,\mathcal{I}(t')}^{=\mathcal{I}\eta}\,;\,\mathcal{I}^2(\alpha^{-1})\,;\,\mu \;=\; \eta\,;\,\mathcal{I}\alpha\,;\,\mathcal{I}(\alpha^{-1})\,;\,\mathcal{I}\eta\,;\,\mu \;=\; \eta$$

$$\theta\,;\,\eta\,;\,\mathcal{I}(\theta^{-1})\,;\,\mathcal{I}(\eta)\,;\,\mu \;=\; \lambda\,;\,\theta^{-1}\,;\,\eta\,;\,\mathcal{I}(\eta)\,;\,\mu \;=\; \eta \qquad \textbf{(for } \theta\in\{\lambda,\rho\}\textbf{)}$$

■

3. Compact Closed Structure

3.1 Compact closedness

Definition 3.1 — Compact closedness. A symmetric monoidal category $(\mathscr{C}, \otimes, I, \sigma)$ is **compact closed** if every object $A \in \mathscr{C}$, regarded as a morphism in the bicategorical delooping of $\mathscr{C}$, has an adjoint $A^* \in \mathscr{C}$ (called *dual object*). Concretely, this means that for every $A \in \mathscr{C}$, A^* comes equipped with a *unit* $i_A \colon I \longrightarrow A \otimes A^*$ and a *counit* $e_A \colon A^* \otimes A \longrightarrow I$ such that

$$A \xrightarrow{\lambda_A^{-1}} I \otimes A \xrightarrow{i_A \otimes 1} (A \otimes A^*) \otimes A \xrightarrow{\alpha_{A,A^*,A}} A \otimes (A^* \otimes A) \xrightarrow{1 \otimes e_A} A \otimes I \xrightarrow{\rho_A} A = \mathrm{id}_A$$

$$A^* \xrightarrow{\rho_{A^*}^{-1}} A^* \otimes I \xrightarrow{1 \otimes i_A} A^* \otimes (A \otimes A^*) \xrightarrow{\alpha_{A^*,A,A^*}^{-1}} (A^* \otimes A) \otimes A^* \xrightarrow{e_A \otimes 1} I \otimes A^* \xrightarrow{\lambda_A} A^* = \mathrm{id}_{A^*}$$

In this section, we will prove that the category $\mathcal{IDL}at$ is compact closed, but for co/units that may not be the most intuitive at the outset. Indeed, drawing upon the fact that the category **Rel** of sets and relations, the Kleisli category of the downset monad over posets, and the category of quantale enriched profunctors (see [KL80; MZ18]) are compact closed, one may be tempted to try to show, in a similar fashion, that so is the case of $\mathcal{IDL}at$ with the following units $i_A \colon \Sigma \longrightarrow\!\!\!\!\circ\!\!\!\rightarrow A \otimes A^\circ$:

$$i_A(\bot) = \downarrow\{a \otimes a' \mid a \leq a' \text{ in } A\} \quad , \quad i_A(\top) = A \otimes A^\circ$$

and counits $e_A \colon A^\circ \otimes A \longrightarrow\!\!\!\!\circ\!\!\!\rightarrow \Sigma$:

$$e_A(a' \otimes a) = \ \downarrow(a' \leq a) \ \ .$$

This turns out *not* to be true unfortunately, because the tensor unit – in our case Σ – is very different from the singleton tensor units of the aforementioned cases. However, $\mathcal{IDL}at$ can still be shown to be compact closed for slightly more intricate co/units, about which we will now provide necessary and sufficient conditions.

First, let us give the following easy charaterisation of ideals in a semilattice:

Proposition 3.1 The ideals of a join semilattice A are exactly the subsets of the form $\downarrow D$, for some $D \subseteq A$ closed under joins.

Proof
Every ideal is closed under joins and equal to its downward closure. Conversely, every subset of the form $\downarrow D$ is clearly down-closed and closed under joins:

$$\begin{cases} d_1 \leq d_1' \in D \\ d_2 \leq d_2' \in D \end{cases} \quad \Longrightarrow \quad d_1 \vee d_2 \leq d_1' \vee d_2' \in D$$

■

This enables us to write $i_A(\bot)$ in the form $\downarrow D$, and give necessary and sufficient conditions on D:

Lemma 3.1 — Necessary and sufficient conditions on co/units. $\mathcal{IDL}at$ is compact closed with units $i_A \colon \Sigma \longrightarrow\!\!\!\!\circ\!\!\!\rightarrow A \otimes A^\circ$ and counits $e_A \colon A^\circ \otimes A \longrightarrow\!\!\!\!\circ\!\!\!\rightarrow \Sigma$ if and only if $i_A(\bot) := \downarrow D$ for some $\vee$-closed $D \subseteq A \otimes A^\circ$ such that for all $a_0 \in A$:

$$\forall \bigwedge_i a_i' \otimes a_i'' \in D, \quad \bigwedge_{j;\ e(a_j'' \otimes a_0) = \{\bot\}} a_j' \leq a_0 \qquad (3.1a)$$

$$\exists \bigwedge_i a_i' \otimes a_i'' \in D; \quad \bigwedge_{j;\ e(a_j'' \otimes a_0) = \{\bot\}} a_j' = a_0 \qquad (3.1b)$$

$$\forall \bigwedge_i a_i' \otimes a_i'' \in D, \quad a_0 \leq \bigvee_{j;\ e(a_0 \otimes a_j') = \{\bot\}} a_j'' \qquad (3.2a)$$

$$\exists \bigwedge_i a_i' \otimes a_i'' \in D; \quad a_0 = \bigvee_{j;\ e(a_0 \otimes a_j') = \{\bot\}} a_j'' \qquad (3.2b)$$

Proof
First, the adjointness conditions for every $A \in \mathcal{IDLat}$ amount to:

$$
\begin{aligned}
& A \xrightarrow{\lambda^{-1}} \Sigma \otimes A \xrightarrow{\eta} \mathcal{I}(\Sigma \otimes A) \xrightarrow{\mathcal{I}(i \otimes \eta)} \mathcal{I}(\mathcal{I}(A \otimes A^\circ) \otimes \mathcal{I}A) \xrightarrow{\mathcal{I}(\mu\, \mathcal{I}(t')\, t)} \mathcal{I}^2((A \otimes A^\circ) \otimes A) \xrightarrow{\mu} \mathcal{I}((A \otimes A^\circ) \otimes A) \\
& \quad \xrightarrow{\mathcal{I}(\alpha)} \mathcal{I}(A \otimes (A^\circ \otimes A)) \xrightarrow{\mathcal{I}(\eta)} \mathcal{I}^2(A \otimes (A^\circ \otimes A)) \xrightarrow{\mu} \mathcal{I}(A \otimes (A^\circ \otimes A)) \\
& \quad \xrightarrow{\mathcal{I}(\eta \otimes e)} \mathcal{I}(\mathcal{I}A \otimes \mathcal{I}\Sigma) \xrightarrow{\mathcal{I}(\mu\, \mathcal{I}(t')\, t)} \mathcal{I}^2(A \otimes \Sigma) \xrightarrow{\mu} \mathcal{I}(A \otimes \Sigma) \\
& \quad \xrightarrow{\mathcal{I}(\rho)} \mathcal{I}A \xrightarrow{\mathcal{I}(\eta)} \mathcal{I}^2 A \xrightarrow{\mu} \mathcal{I}A \\
& \stackrel{?}{=} \quad \eta_A
\end{aligned}
$$

and

$$
\begin{aligned}
& A^\circ \xrightarrow{\rho^{-1}} A^\circ \otimes \Sigma \xrightarrow{\eta} \mathcal{I}(A^\circ \otimes \Sigma) \xrightarrow{\mathcal{I}(\eta \otimes i)} \mathcal{I}(\mathcal{I}(A^\circ) \otimes \mathcal{I}(A \otimes A^\circ)) \xrightarrow{\mathcal{I}(\mu\, \mathcal{I}(t)\, t')} \mathcal{I}^2(A^\circ \otimes (A \otimes A^\circ)) \xrightarrow{\mu} \mathcal{I}(A^\circ \otimes (A \otimes A^\circ)) \\
& \quad \xrightarrow{\mathcal{I}(\alpha^{-1})} \mathcal{I}((A^\circ \otimes A) \otimes A^\circ) \xrightarrow{\mathcal{I}(\eta)} \mathcal{I}^2((A^\circ \otimes A) \otimes A^\circ) \xrightarrow{\mu} \mathcal{I}((A^\circ \otimes A) \otimes A^\circ) \\
& \quad \xrightarrow{\mathcal{I}(e \otimes \eta)} \mathcal{I}(\mathcal{I}(\Sigma) \otimes \mathcal{I}(A^\circ)) \xrightarrow{\mathcal{I}(\mu\, \mathcal{I}(t)\, t')} \mathcal{I}^2(\Sigma \otimes A^\circ) \xrightarrow{\mu} \mathcal{I}(\Sigma \otimes A^\circ) \\
& \quad \xrightarrow{\mathcal{I}(\lambda)} \mathcal{I}(A^\circ) \xrightarrow{\mathcal{I}(\eta)} \mathcal{I}^2(A^\circ) \xrightarrow{\mu} \mathcal{I}(A^\circ) \\
& \stackrel{?}{=} \quad \eta_{A^\circ}
\end{aligned}
$$

Due to $\mathcal{I}$ being a monad, η, μ being natural, $\begin{cases} t' \,;\, \mathcal{I}(\rho) = \rho \\ t \,;\, \mathcal{I}(\lambda) = \lambda \end{cases}$ and $\begin{cases} 1 \otimes \eta \,;\, t = \eta \\ \eta \otimes 1 \,;\, t' = \eta \end{cases}$, this simplifies to:

$$
\begin{aligned}
& A \xrightarrow{\phi_A} \mathcal{I}A \\
:= \; & A \xrightarrow{\lambda^{-1}} \Sigma \otimes A \xrightarrow{i \otimes 1} \mathcal{I}(A \otimes A^\circ) \otimes A \xrightarrow{t'} \mathcal{I}((A \otimes A^\circ) \otimes A) \\
& \xrightarrow{\mathcal{I}(\alpha)} \mathcal{I}(A \otimes (A^\circ \otimes A)) \xrightarrow{\mathcal{I}(\eta \otimes e)} \mathcal{I}(\mathcal{I}A \otimes \mathcal{I}\Sigma) \xrightarrow{\mathcal{I}(t)} \mathcal{I}^2(\mathcal{I}A \otimes \Sigma) \\
& \xrightarrow{\mu} \mathcal{I}(\mathcal{I}A \otimes \Sigma) \xrightarrow{\mathcal{I}(\rho)} \mathcal{I}^2(A) \xrightarrow{\mu} \mathcal{I}A \\
& \stackrel{?}{=} \quad \eta_A
\end{aligned}
$$

and

$$
\begin{aligned}
& A^\circ \xrightarrow{\psi_{A^\circ}} \mathcal{I}(A^\circ) \\
:= \; & A^\circ \xrightarrow{\rho^{-1}} A^\circ \otimes \Sigma \xrightarrow{1 \otimes i} A^\circ \otimes \mathcal{I}(A \otimes A^\circ) \xrightarrow{t} \mathcal{I}(A^\circ \otimes (A \otimes A^\circ)) \\
& \xrightarrow{\mathcal{I}(\alpha^{-1})} \mathcal{I}((A^\circ \otimes A) \otimes A^\circ) \xrightarrow{\mathcal{I}(e \otimes \eta)} \mathcal{I}(\mathcal{I}(\Sigma) \otimes \mathcal{I}(A^\circ)) \xrightarrow{\mathcal{I}(t')} \mathcal{I}^2(\Sigma \otimes \mathcal{I}(A^\circ)) \\
& \xrightarrow{\mu} \mathcal{I}(\Sigma \otimes \mathcal{I}(A^\circ)) \xrightarrow{\mathcal{I}(\lambda)} \mathcal{I}^2(A^\circ) \xrightarrow{\mu} \mathcal{I}(A^\circ) \\
& \stackrel{?}{=} \quad \eta_{A^\circ}
\end{aligned}
$$

Elementwise, this is equally to show that for all $a_0 \in A$:

$$
\begin{aligned}
\phi_A(a_0) &= \{\, a \in A \mid \exists \delta \in \mathcal{I}A \otimes \Sigma;\ \exists \hat{a} \in i(\bot);\ \exists d \leq \hat{a} \otimes a_0;\ \downarrow \delta \leq t(\eta \otimes e)\alpha(d);\ \downarrow a \leq \rho\delta \,\} \\
&\stackrel{?}{=} \eta_A(a_0) = \downarrow a_0 \\
\psi_{A^\circ}(a_0) &= \{\, a \in A^\circ \mid \exists \delta \in \Sigma \otimes \mathcal{I}(A^\circ);\ \exists \hat{a} \in i(\bot);\ \exists d \leq a_0 \otimes \hat{a};\ \downarrow \delta \leq t'(e \otimes \eta)\alpha^{-1}(d);\ \uparrow a \leq \lambda\delta \,\} \\
&\stackrel{?}{=} \eta_{A^\circ}(a_0) = \uparrow a_0
\end{aligned}
$$

We will now hone in on the first equality, the second one being analogous. By Proposition 3.1, the ideal $i(\bot)$ can be written as $\downarrow D$, for some $D \subseteq A$ closed under joins. It comes that:

$$
\begin{aligned}
& \phi_A(a_0) \subseteq \downarrow a_0 \\
\iff\ & \forall a \in A, a \in \phi_A(a_0) \Rightarrow a \leq a_0 \\
\iff\ & \forall a \in A, \delta \in \mathcal{I}A \otimes \Sigma,\ \hat{a} \leq \textstyle\bigwedge_i a'_i \otimes a''_i \in D,\ d \leq \hat{a} \otimes a_0, \\
& \begin{cases} \downarrow \delta \leq t(\eta \otimes e)\alpha(d) \\ \downarrow a \leq \rho\delta \end{cases} \Rightarrow\ a \leq a_0 \\
\Longleftarrow\ & \forall a \in A, \delta \in \mathcal{I}A \otimes \Sigma,\ \textstyle\bigwedge_i a'_i \otimes a''_i \in D, \\
& \begin{cases} \downarrow \delta \leq \bigwedge_i \underbrace{t(\eta \otimes e)(a'_i \otimes (a''_i \otimes a_0))}_{=\downarrow a'_i \otimes e(a''_i \otimes a_0)} = \bigwedge_i \downarrow \left\{ \downarrow a'_i \otimes o_i \right\}_{o_i \in e(a''_i \otimes a_0)} \\ \downarrow a \leq \rho\delta \end{cases} \Rightarrow\ a \leq a_0
\end{aligned}
$$

But when it comes to $\bigwedge_i \downarrow \left\{ \downarrow a'_i \otimes o_i \right\}_{o_i \in e(a''_i \otimes a_0)}$, there are two alternatives for each $e(a''_i \otimes a_0) \in \mathcal{I}(\Sigma)$:

- either $e(a''_i \otimes a_0) = \Sigma$, in which case $\left\{ \downarrow a'_i \otimes o_i \right\}_{o_i \in e(a''_i \otimes a_0)}$ contains the top element and $\downarrow \left\{ \downarrow a'_i \otimes o_i \right\}_{o_i \in e(a''_i \otimes a_0)}$ is the top element of $\mathcal{I}(\mathcal{I}A \otimes \Sigma)$
- or $e(a''_i \otimes a_0) = \{\bot\}$, in which case $\downarrow \left\{ \downarrow a'_i \otimes o_i \right\}_{o_i \in e(a''_i \otimes a_0)} = \downarrow(\downarrow a'_i \otimes \bot)$

Therefore

$$
\begin{aligned}
& \phi_A(a_0) \subseteq \downarrow a_0 \\
\Longleftarrow\ & \forall a \in A, \delta \in \mathcal{I}A \otimes \Sigma,\ \textstyle\bigwedge_i a'_i \otimes a''_i \in D, \\
& \begin{cases} \downarrow \delta \leq \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} \downarrow (\downarrow a'_j \otimes \bot) = \downarrow \left(\left(\downarrow \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \right) \otimes \bot \right) \\ \downarrow a \leq \rho\delta \leq \downarrow \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \end{cases} \Rightarrow\ a \leq a_0 \\
\Longleftarrow\ & \forall a \in A,\ \textstyle\bigwedge_i a'_i \otimes a''_i \in D, \quad a \leq \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \Rightarrow\ a \leq a_0 \\
\iff\ & \forall \textstyle\bigwedge_i a'_i \otimes a''_i \in D, \quad \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \leq a_0
\end{aligned}
$$

which shows that eq. (3.1a) is a sufficient condition. It is also necessary: if $\phi_A(a_0) \subseteq \downarrow a_0$, then for all $\bigwedge_i a'_i \otimes a''_i \in D$, by setting

$$
\begin{aligned}
d &:= \left(\textstyle\bigwedge_i a'_i \otimes a''_i \right) \otimes a_0 \\
\delta &:= \downarrow \left(\textstyle\bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \right) \otimes \bot
\end{aligned}
$$

it comes that $a := \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \in \phi_A(a_0)$, since

$$
\begin{cases} \downarrow \delta & \leq \downarrow \left(\downarrow \left(\bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \right) \otimes \bot \right) = \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} \downarrow(\downarrow a'_j \otimes \bot) = \bigwedge_i \downarrow \left\{ \downarrow a'_i \otimes o_i \right\}_{o_i \in e(a''_i \otimes a_0)} = t(\eta \otimes e)\alpha(d) \\ \downarrow a & \leq \downarrow \left(\bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \right) = \rho\delta \end{cases}
$$

hence $\bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j = a \in \downarrow a_0$.

Finally, the existence of an element $\bigwedge_i a'_i \otimes a''_i \in D$ such that $a_0 \leq \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j$ is equivalent to $\phi_A(a_0) \supseteq \downarrow a_0$, *i.e.* $\phi_A(a_0) \ni a_0$ (as $\phi_A(a_0)$ is down-closed):

- It is sufficient: if $a_0 \leq \bigwedge_{j;\, e(a''_j \otimes a_0)=\{\perp\}} a'_j$, then by putting

$$
\begin{aligned}
d \;&:=\; \left(\bigwedge\nolimits_i a'_i \otimes a''_i\right) \otimes a_0 \\
\delta \;&:=\; \downarrow a_0 \otimes \perp
\end{aligned}
$$

  we have $\downarrow a_0 \leq \rho\delta$ and $\downarrow \delta \leq \downarrow \left(\downarrow \left(\bigwedge_{j;\, e(a''_j \otimes a_0)=\{\perp\}} a'_j\right) \otimes \perp\right) = t(\eta \otimes e)\alpha(d)$

- It is necessary: if there exist $\delta \in \mathcal{I}A \otimes \Sigma$, $\hat{a} \leq \bigwedge_i a'_i \otimes a''_i \in D$, $d \leq \hat{a} \otimes a_0$ such that

$$
\begin{cases}
\downarrow \delta \leq t(\eta \otimes e)\alpha(d) \leq \bigwedge_i \downarrow \left\{\downarrow a'_i \otimes o_i\right\}_{o_i \in e(a''_i \otimes a_0)} = \downarrow \left(\downarrow \left(\bigwedge_{j;\, e(a''_j \otimes a_0)=\{\perp\}} a'_j\right) \otimes \perp\right) \\
\downarrow a_0 \leq \rho\delta \leq \downarrow \left(\bigwedge_{j;\, e(a''_j \otimes a_0)=\{\perp\}} a'_j\right)
\end{cases}
$$

  then the result follows.

Therefore, eqs. (3.1a) and (3.1b) are equivalent to $\phi_A(a_0) = \downarrow a_0$, and we can analogously show that eqs. (3.2a) and (3.2b) are equivalent to $\psi_A(a_0) = \uparrow a_0$.

■

NB If A satisfies the Descending Chain Condition (DCC), so that every $a_0 \in A$ is a join of $\vee$-irreductibles and meet of $\wedge$-irreducibles (which correspond exactly to $\vee$-primes and $\wedge$-primes respectively by distributivity), then for every subset $D \subseteq A \otimes A^\circ$ satisfying eqs. (3.1a), (3.1b), (3.2a) and (3.2b), the $\vee$-closure $D^\vee$ still satisfies eqs. (3.1a), (3.1b), (3.2a) and (3.2b).

This now brings us to our main result:

Theorem 3.2 The category $\mathcal{IDL}at$ is compact closed, with

- the counits $e_A \colon A^\circ \otimes A \multimap \Sigma$ given by

$$e_A(a' \otimes a) = \downarrow(a' \leq a)$$

- the units $i_A \colon \Sigma \multimap A \otimes A^\circ$ given by

$$i_A(\perp) = \downarrow\left(\{\perp \otimes a \wedge a \otimes \top \mid a \in A\}^\vee\right) \;, \quad i_A(\top) = A \otimes A^\circ$$

■

Proof

It suffices to show that $D := \{\perp \otimes a \wedge a \otimes \top \mid a \in A\}^\vee$ satisfies the necessary and sufficient conditions of *Lemma* 3.1. Let $A \ni a_0 \neq \top$ and $D \ni \bigvee\limits_{i \in I} \left(\perp \otimes a_i \,\wedge\, a_i \otimes \perp\right) = \bigwedge\limits_{J \subseteq I} \left(\underbrace{\left(\bigvee\limits_{j \in J} a_j\right)}_{\text{denoted by } a'_J} \otimes \underbrace{\left(\bigwedge\limits_{k \notin J} a_k\right)}_{\text{denoted by } a''_J}\right)$

by distributivity. To prove eq. (3.1a), we show that

$$\bigwedge_{J';\, a''_{J'} \not\leq a_0} a'_{J'} \leq a_0 \tag{3.3}$$

The index set I can be partitioned:

$$I = \hat{I} \sqcup \check{I}$$

where $\hat{I} := \{k \in I \mid a_k \leq a_0\}$ and $\check{I} := \{k \in I \mid a_k \not\leq a_0\}$. Then $\bigvee\limits_{k \in \hat{I}} a_k \leq a_0$, and:

- `Case 1`: if $\bigwedge_{k\in\check{I}} a_k \not\leq a_0$, then eq. (3.3) follows, since $\begin{cases} a'_{\hat{I}} = \bigvee_{k\in\hat{I}} a_k \leq a_0 \\ a''_{\hat{I}} = \bigwedge_{k\in\check{I}} a_k \not\leq a_0 \end{cases}$ hence $\bigwedge_{J';a''_{J'}\not\leq a_0} a'_{J'} \leq a'_{\hat{I}} \leq a_0$
- `Case 2`: else if

$$\bigwedge_{k\in\check{I}} a_k \leq a_0 \tag{3.4}$$

As $\begin{cases} \forall k \in \check{I}, a_k = \bigwedge\{a_k\} \not\leq a_0 \\ \bigwedge_{k\in\check{I}} a_k = \bigwedge\{a_k \mid k \in \check{I}\} \leq a_0 \end{cases}$ it comes that

$$\Gamma := \Big\{\emptyset \neq S \subsetneq \check{I} \mid S \text{ maximal such that } a''_{I\setminus S} := \bigwedge_{k\in S} a_k \not\leq a_0\Big\} \neq \emptyset$$

Let us show that

$$\bigwedge_{S\in\Gamma} a'_{I\setminus S} \leq a_0 \tag{3.5}$$

Indeed, since

$$a'_{I\setminus S} = \bigvee_{k\in I\setminus S} a_k = \bigvee_{k\in\hat{I}\sqcup(\check{I}\setminus S)} a_k = a'_{\hat{I}} \vee \bigvee_{k\in\check{I}\setminus S} a_k$$

we have, by setting $\Gamma := \{S_1, \ldots, S_m\}$ and $a_* := a'_{\hat{I}}$:

$$\begin{aligned} &\bigvee_{k_1\in\{*\}\cup\check{I}\setminus S_1,\,\ldots,\,k_m\in\{*\}\cup\check{I}\setminus S_m} \bigwedge_{l=1}^{m} a_{k_l} = \bigwedge_{S\in\{S_1,\ldots,S_m\}} \Big(a'_{\hat{I}} \vee \bigvee_{k\in\check{I}\setminus S} a_k\Big) = \bigwedge_{S\in\Gamma} a'_{I\setminus S} \leq a_0 \\ \iff\quad &\forall k_1 \in \{*\} \cup \check{I} \setminus S_1,\, \ldots,\, k_m \in \{*\} \cup \check{I} \setminus S_m, \qquad \bigwedge_{l=1}^{m} a_{k_l} \leq a_0 \end{aligned}$$

And the last assertion does hold:

  - `Case 2a`: if one of the k_i's is equal to $*$, then we are done, as

$$\bigwedge_{l=1}^{m} a_{k_l} \leq a_* \leq a_0$$

  - `Case 2b`: else, suppose by contradiction that $\bigwedge_{l=1}^{m} a_{k_l} \not\leq a_0$. Then

$$\begin{cases} \emptyset \neq \{k_1, \ldots, k_m\} \subsetneq \check{I} & \text{(by assumption and eq. (3.4))} \\ \bigwedge_{k\in\{k_1,\ldots,k_m\}} a_k \not\leq a_0 \end{cases}$$

    Therefore, by maximality of the elements of Γ, there exists $S_i \in \Gamma$ such that $\{k_1, \ldots, k_m\} \subseteq S_i$. But this contradicts the definition of k_i.

As a consequence, eq. (3.3) directly follows from eq. (3.5), since

$$\bigwedge_{J';a''_{J'}\not\leq a_0} a'_{J'} \leq \bigwedge_{S\in\Gamma} a'_{I\setminus S} \leq a_0$$

Finally, the condition eq. (3.1b) is satisfied owing to $\bot \otimes a_0 \wedge a_0 \otimes \top \in D$, and eqs. (3.2a) and (3.2b) can be shown analogously, by symmetry.

■

4. Model of Linear Logic

4.1 Categorical models

Recall that a model of the multiplicative exponential fragment of classical linear logic (CLL) is given by [Bar91; Bie95; Mel03; Sch]:

1. a $*$-autonomous category $(\mathscr{C}, \otimes, \multimap, (-)^{\perp})$, to model the *multiplicatives* with the SMC structure and the *linear negation* with the dualisation operation $(-)^{\perp}$
2. which has finite products $\&$ (and thus coproducts $\oplus$, induced by dualisation) to model the *additives*
3. and is equipped with a linear exponential comonad $!$ (and thus a linear exponential monad $?$, by dualisation) – that is: a monoidal monad $!$ that lifts the tensor $\otimes$ to a coproduct in the category of $!$-algebras – to model the *exponentials*.

In a $*$-autonomous category $\mathscr{C}$, the monoidal structure of multiplicative conjunction $(\otimes, 1)$ uniquely determines that of multiplicative disjunction $(\invamp, \perp)$ by "de Morgan duality" (and vice versa): $A \invamp B \cong (A^{\perp} \otimes B^{\perp})^{\perp}$. For each object $A \in \mathscr{C}$, $A^{\perp}$ is referred to as its *dual*, in the sense that it can be seen as a weak form of adjoint in the bicategorical delooping of $\mathscr{C}$ where the co/units:

$$i_A \colon 1 \longrightarrow A \invamp A^{\perp} \qquad e_A \colon A^{\perp} \otimes A \longrightarrow \perp$$

mix both tensor products. A special case of significant historical importance is when the model is *degenerate*, that is, when the mutliplicatives ($\otimes$ and $\invamp$) coincide, and so do the additives and the exponentials. In this case, the dual objects are actual adjoints in the delooping of $\mathscr{C}$, and the category $\mathscr{C}$ is **compact closed** [KL80].

4.2 Additives

It is well-known that

> **Proposition — [Szi], Proposition 2.2.** For every category $\mathscr{C}$ and monad $T\colon \mathscr{C} \longrightarrow \mathscr{C}$, if $\mathscr{C}$ has coproducts, then so does $\mathcal{K}\ell(T)$, by post composing the coprojections by the monad unit.

As a result, as $\mathcal{IDL}at = \mathcal{K}\ell(\mathscr{I})$ is compact closed and $\boldsymbol{\mathcal{MDL}at}$ has coproducts (see Proposition D.1):

Corollary 4.1 $\mathcal{IDL}at$ has biproducts.

Proof
Houston showed in [Hou08] that a compact closed category has finite biproducts as soon as it has finite coproducts (this result was generalised by Garner and Schäppi in [GS16]). ■

4.3 Exponentials

We now go on to exhibit a linear exponential monad on $\mathcal{IDL}at$ modeling the why-not modality. As $\mathcal{IDL}at$ is compact closed, such a monad will give rise to a linear exponential comonad, by strong duality.

4.3.1 Monadicity of commutative monoids

Given a symmetric monoidal category $(\mathscr{C}, \invamp, \perp)$, a convenient and common way to concoct a linear exponential monad on $\mathscr{C}$ is to consider whether the forgeful functor $\mathscr{U}\colon \mathrm{CMon}_{\invamp}(\mathscr{C}) \longrightarrow \mathscr{C}$ from the category $\mathrm{CMon}_{\invamp}(\mathscr{C})$ of commutative $\invamp$-monoids to $\mathscr{C}$ is *monadic* – meaning that it has a left adjoint

$F \underset{\eta}{\overset{\epsilon}{\dashv}} \mathscr{U}$ and the canonical comparison functor[1] $K^M\colon \mathrm{CMon}_{⅋}(\mathscr{C}) \longrightarrow \mathcal{EM}(M)$ for the induced monad $M := \mathscr{U}F$ is an equivalence. In such a case, it is well-known that the induced monad M is linear exponential, owing to the fact that ⅋ can be lifted to $\mathrm{CMon}_{⅋}(\mathscr{C}) \simeq \mathcal{EM}(M)$ (see for example [JS93]), where it becomes a coproduct, in this manner:

$$
\begin{aligned}
&(A_1,\ m_1\colon A_1 ⅋ A_1 \longrightarrow A_1,\ e\colon \bot \longrightarrow A_1) ⅋ (A_2,\ m_2\colon A_2 ⅋ A_2 \longrightarrow A_2,\ e_2\colon \bot \longrightarrow A_2)\\
:= \Big(&A_1 ⅋ A_2,\ m := (A_1 ⅋ A_2) ⅋ (A_1 ⅋ A_2)\\
&\qquad \cong A_1 ⅋ (A_2 ⅋ A_1) ⅋ A_2 \xrightarrow{1 ⅋ \sigma ⅋ 1} A_1 ⅋ (A_1 ⅋ A_2) ⅋ A_2\\
&\qquad \cong (A_1 ⅋ A_1) ⅋ (A_2 ⅋ A_2) \xrightarrow{m_1 ⅋ m_2} A_1 ⅋ A_2\\
&\quad e := \bot \cong \bot ⅋ \bot \xrightarrow{e_1 ⅋ e_2} A_1 ⅋ A_2\Big)
\end{aligned}
$$

the coprojections being given by

$$
\kappa_1\colon A_1 \cong A_1 ⅋ \bot \xrightarrow{1 ⅋ e_2} A_1 ⅋ A_2 \qquad\qquad \kappa_2\colon A_2 \cong \bot ⅋ A_2 \xrightarrow{e_1 ⅋ 1} A_1 ⅋ A_2
$$

Now, Beck's acclaimed monadicity theorem [Bec] characterising monadicity:

Theorem 4.2 — Beck's monadicity theorem. A functor $\mathscr{U} : \mathscr{D} \longrightarrow \mathscr{C}$ is monadic iff

1. $\mathscr{U}$ has a left adjoint
2. $\mathscr{U}$ reflects isomorphisms
3. $\mathscr{C}$ has and $\mathscr{U}$ preserves coequalisers of $\mathscr{U}$-split pairs, which are those parallel pairs $f, g\colon A \longrightarrow B \in \mathscr{D}$ sent by $\mathscr{U}$ to a pair $\mathscr{U}f$, $\mathscr{U}g$ having a split coequaliser in $\mathscr{C}$, *i.e.* such that there exists a cocone $A \overset{\mathscr{U}f}{\underset{\mathscr{U}g}{\rightrightarrows}} B \xrightarrow{h} C$ such that the morphism $(\mathscr{U}f, h)\colon \mathscr{U}g \longrightarrow h$ has a section in the arrow category of $\mathscr{C}$.

does so by convieniently framing it as conditions on $\mathscr{U}$, the last two of which happen to be satisfied by the forgetful functor $\mathscr{U} :\ \mathrm{CMon}_{⅋}(\mathscr{C}) \longrightarrow \mathscr{C}$ [Hyl+06; Kel80; Lan78]. So the question of monadicity of $\mathscr{U}$ reduces to the existence of a left adjoint.

4.3.2 Extension to the Kleisli category

But that is not all: in our case, $(\mathscr{C}, ⅋) := (\mathcal{IDL}at := \mathcal{K}\ell(\mathscr{I}), \otimes)$ is a Kleisli category, and it might seem a bit tricky, at first glance, to directly work there. The traditional approach to go about proving that we have a monad $? := \mathscr{U}F$ on a Kleisli category $\mathcal{K}\ell(T)$ – where $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ – is to

1. first construct it in the base category $\mathscr{A}$
2. before extending it to the Kleisli category $\mathcal{K}\ell(T)$, by resorting to the following theorem by Beck:

Theorem 4.3 — Beck [Bec69]. The following are equivalent:

- an extension of $?$ to a monad on $\mathcal{K}\ell(T)$
- a lifting of T to a monad on $\mathcal{EM}(?)$

[1] defined on objects by $K^M(C) := \mathscr{U}(\epsilon_C) : \mathscr{U}F\mathscr{U}C \longrightarrow \mathscr{U}C$

- a distributive law of $?$ over T, *i.e.* a natural transformation $\lambda\colon ?T \Rightarrow T?$ subject to coherence conditions (see [Bec69])

This theorem comes in handy because lifting T to $\mathcal{EM}(?)$ is usually easier, in practice, than extending $?$ to $\mathcal{K}\ell(T)$.

In our case where $\mathscr{A} := \boldsymbol{\mathcal{MDL}at}$, we are then brought to construct a left adjoint to the forgetful functor $\mathscr{U}'\colon \mathrm{CMon}_{\otimes}(\boldsymbol{\mathcal{MDL}at}) \longrightarrow \boldsymbol{\mathcal{MDL}at}$. This can be attempted

- using Freyd's adjoint functor theorem [FŠ90], which amounts to proving that for all $A \in \boldsymbol{\mathcal{MDL}at}$, the comma category $A \downarrow \mathscr{U}'$ has an initial object, by showing that $\mathrm{CMon}_{\otimes}(\boldsymbol{\mathcal{MDL}at})$ is complete (which implies that the comma category is complete as well) and that there exists a weakly initial set of objects in the comma category. A sufficient condition given by Marty in [Mar, Proposition 1.2.14.] for $\mathrm{CMon}_{\otimes}(\boldsymbol{\mathcal{MDL}at})$ to be complete, is that $\boldsymbol{\mathcal{MDL}at}$ be itself complete, under the (itself far from obvious) assumption that it is monoidal closed. But this does not hold, as seen in Proposition 2.3, since $\boldsymbol{\mathcal{MDL}at}$ does not have equalisers.
- by constructing the adjoint by hand, in the style of [Lan78, Theorem VII.3.2]. But again, sufficient conditions we find ourselves wishing for are
  - the tensor preserving countable colimits (which would be true if $\boldsymbol{\mathcal{MDL}at}$ were monoidal closed, but – to our knowledge – this is an open problem posed by Barr [Bara; Barb]).
  - $\boldsymbol{\mathcal{MDL}at}$ having finite coequalisers, which is not true by Proposition 2.3.

However, all hope is not lost: free commutative monoids can be constructed explicitly in $\boldsymbol{\mathcal{IDL}at}$.

4.3.3 Explicit construction

For the aforementioned reasons, when free commutative monoids exist, the resulting monad is linear exponential. Provided that $\mathscr{C}$ has countable colimits and the tensor preserves them, the free commutative monoid on A can be constructed explicitly by the well-known exponential formula (see for example [MTT]):

$$\sum_{n\geq 0} A^{\otimes n}/\mathfrak{S}_n$$

where each summand is the coequaliser of the $n!$ permutations $A^{\otimes n} \longrightarrow A^{\otimes n}$. Such a coequaliser does exist in $\boldsymbol{\mathcal{MDL}at}$, by taking the coequaliser in the bicomplete category $\boldsymbol{\mathcal{DL}at}$ of distributive lattices and lattice morphisms (preserving finite meets and joins), which is enabled by the fact that permutations preserve joins too. It remains to be shown that:

- For every $n \geq 0$, the object $A^{\otimes n}/\mathfrak{S}_n \in \boldsymbol{\mathcal{DL}at}$ is a coequaliser in $\boldsymbol{\mathcal{MDL}at}$. Given that $\boldsymbol{\mathcal{DL}at}$ is not a coreflective subcategory of $\boldsymbol{\mathcal{MDL}at}$ (Σ is the initial object in $\boldsymbol{\mathcal{DL}at}$ but not in $\boldsymbol{\mathcal{MDL}at}$), the inclusion functor does not preserve colimits *a priori*. But we can still show by hand that $A^{\otimes n}/\mathfrak{S}_n \in \boldsymbol{\mathcal{DL}at}$ enjoys the coequaliser universal property in $\boldsymbol{\mathcal{MDL}at}$ (Lemma D.1, proof in appendix):

  Lemma — D.1 – Coequalisers of identity and permutations of n-th tensor powers.
  Let $A \in \boldsymbol{\mathcal{MDL}at}$, $n \in \mathbb{N}$, $k \in [\![1, n!]\!]$ and $\sigma_1, \ldots, \sigma_k\colon A^{\otimes n} \longrightarrow A^{\otimes n} \in \boldsymbol{\mathcal{MDL}at}$ be permutations of the n-th tensor power $A^{\otimes n}$. Then the coequaliser of the parallel morphisms

  $$\sigma_1, \ldots, \sigma_k\colon A^{\otimes n} \longrightarrow A^{\otimes n}$$

  exists in $\boldsymbol{\mathcal{MDL}at}$ by lifting it from $\boldsymbol{\mathcal{DL}at}$.

- The said coequalisers can be lifted to $\mathcal{IDLat}$, as the free algebra functor $\mathcal{MDLat} \longrightarrow \mathcal{K}\ell(\mathcal{I})$ is a left adjoint, and thus preserves colimits.

As a consequence, by Corollary 4.1, free commutative monoids can be constructed explicitly in $\mathcal{IDLat}$, as the tensor preserves colimits due to $\mathcal{IDLat}$ being monoidal closed, and it follows that:

Corollary 4.4 The linear exponential free commutative monoids monad can be defined on $\mathcal{IDLat}$, thereby modeling the why-not modality of linear logic.

Finally, as $\mathcal{IDLat}$ is compact closed:

Corollary 4.5 The why-not and of-course exponential modalities can be modeled in $\mathcal{IDLat}$.

On the whole, putting the previous results together (Theorem 3.2, Corollary 4.1, and Corollary 4.5), we have:

Theorem 4.6 $\mathcal{IDLat} := \mathcal{K}\ell(\mathcal{I})$ is a degenerate model of classical linear logic. ■

4.4 Dualisation operation

As LL connectives come in pairs of De Morgan duals that determine each other, a key step to better understand the model at hand involves describing its dualisation operation (corresponding to linear negation). In every compact closed category, the dual of a morphism $f\colon A \longrightarrow B$ is given by:

$$f^* := B^* \xrightarrow{\rho^{-1}} B^* \otimes \Sigma \xrightarrow{1 \otimes i} B^* \otimes (A \otimes A^*) \xrightarrow{1 \otimes (f \otimes 1)} B^* \otimes (B \otimes A^*) \xrightarrow{\alpha^{-1} ; (e \otimes 1)} \Sigma \otimes A^* \xrightarrow{\lambda} A^* \tag{4.1}$$

in such a way that the assignment $f \longmapsto f^*$ extends to a contravariant involutive functor sometimes referred to as *star-involution* and endowing the category with a $*$-autonomous structure. In $\mathcal{IDLat}$, eq. (4.1) boils down to

$$\begin{aligned} f^* := \quad & \rho^{-1} \,;\, \eta \,;\, \mathcal{I}\big((\eta \otimes i) \,;\, t \,;\, \mathcal{I}(t') \,;\, \mu\big) \,;\, \mu \\ & ;\, \mathcal{I}\Big(\eta \otimes \big((f \otimes i) \,;\, t \,;\, \mathcal{I}(t') \,;\, \mu\big) \,;\, t \,;\, \mathcal{I}(t') \,;\, \mu\Big) \,;\, \mu \\ & ;\, \mathcal{I}\big(\alpha^{-1} \,;\, (e \otimes \eta) \,;\, t \,;\, \mathcal{I}(t') \,;\, \mu\big) \,;\, \mu \,;\, \mathcal{I}(\lambda) \end{aligned}$$

Surprisingly, this messy and intimidating expression can be shown to simplify to the following elegant one:

Theorem 4.7
In $\mathcal{IDLat}$, the dualisation operation $(-)^* = (-)^\perp$ is given on every morphism $f\colon A \multimap B$ by

$$f^\perp \colon \begin{cases} B^\circ & \longrightarrow \mathcal{I}(A^\circ) \\ b & \longmapsto \{a \in A^\circ \mid b \in f(a)\} \end{cases} \tag{4.2}$$

■

We omit the proof, which is quite involved (Lemma 3.1 is used). However, we can easily check that eq. (4.2) statisfies the necessary condition of yielding a morphism in $\mathcal{K}\ell(\mathcal{I})$, *i.e.* that $f^\perp$ is monotone and preserves finite meets, and $f^\perp(b)$ is an ideal for every $b \in B^\circ$.

5. Future work and conclusion

Further directions

This work was a first elementary step towards numerous potential further investigations:

- `LL structure`: Our pedestrian approach of constructing the exponentials leaves a lot to be desired, due to it being *ad hoc* and not lending itself to generalisation. One may want to explore further the aforementioned traditional approaches section 4.3.2, as they are more transferable to other contexts. To do so, a recurrent question is whether $\mathcal{MDL}at$ is monoidal closed – an open problem posed by Barr [Bara; Barb] – and if so, whether we can lift the free commutative monoids monad from $\mathcal{MSL}at$ to $\mathcal{IDL}at$ via a distributive law that would hopefully come from Day convolution for Ind-completion. Besides, the very existence of the "permutation coequalisers" in $\mathcal{MDL}at$ seems to stem from more general and natural considerations: it may have to do with the monadic forgetful functor $\mathscr{U} : \mathcal{DL}at \to \mathcal{MSL}at$ creating these coequalisers (to get the result by Beck's monadicity theorem, one may be tempted to show that they are coequalisers of $\mathscr{U}$-split pairs, but this does not seem to be the case). On another unrelated note, it would be interesting to investigate whether we additionally have a model of differential LL [Ehr16; ER03], and whether we can come up with a form of coherence and/or hypercoherence structure [Bou06].
- `Compact closedness`: Compact closed categories play an important role in Abramsky and Coecke's approach to quantum computation [AC07], and especially dagger compact categories [Sel07]. Given that $\mathbf{Rel}$ and the category **FdHilb** of finite dimensional Hilbert spaces (with the tensor classifying linear maps) are dagger compact closed, a natural question is whether it is the case of $\mathcal{IDL}at$ too. We made several attempts to get a dagger structure (with adjoints and prime ideals, with complements and prime ideals, with minimal elements in the preimage of each ideal, ...) inspired from the situation in $\mathbf{Rel}$ and **FdHilb**, but none were successful. The question remains open.
- `Categorification`: Another improvement would be to categorify the current situation.
  - For example, we have been working in the Σ-enriched setting: the next step would be to try to generalise the current results to the Q-enriched one, for an arbitrary quantale Q, before tackling the $\mathscr{V}$-enriched one, for a general Bénabou cosmos $\mathscr{V}$.
  - A tantalising generalisation would be the following. Let $\mathscr{K}$ be a bicategory of small categories with property-like structure: *e.g.* a distinguished class of co/limits Φ (in our case, we had products $\wedge$) given by a KZ-doctrine, and enriched profunctors preserving these Φ-co/limits. Then, consider the *full* subcategory $\mathscr{K}' \subseteq \mathscr{K}$ comprised of categories having another class Ψ of co/limits commuting with Φ-co/limits (involving a pseudo-distributive law). Can we give sufficient conditions for the tensor $\mathscr{C} \otimes \mathscr{D}$ classifying bihomomorphisms (preserving Φ-co/limits componentwise) to be an object of $\mathscr{K}'$ (which holds in our case for distributive lattices, where Ψ-colimits are coproducts)? And if so, do we still have compact closedness (in the bicategorical sense), by generalising our co/units? A main obstacle is that our results heavily rely on Fraser's theorem for distributive lattices [Fra76, Theorem 2.6], which may not carry over to more general settings.
  - Another line of investigation would involve focusing on categorfying the ideal monad as a KZ-doctrine or a Yoneda structure, and see which results lend themselves to generalisation in this context.

Conclusion

We considered the Kleisli category of the monad of ideals on bounded distributive lattices with meet-preserving maps, and showed that it is compact closed for the tensor product classifying bihomomorphisms. Moreover, it constitutes a model of full classical linear logic.

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Appendix

A. Notations

A.0.1 Prerequisites

We assume familiarity with category theory (categories, functors, natural transformations, adjoints, co/monads, Kleisli and Eilenberg-Moore categories, co/limits, Lawvere theories, monoidal categories, profunctors, bicategories, ...) and Girard's linear logic [Gir87; Mel], even though the most important notions will often be recalled. Good introductions to category theory are [Lan78], [Rie17], and [Lei16] for example. Non-standard notations are introduced when used for the first time, but for convenience, a glossary of notations/abbreviations can be found in here.

General notations

$\cong$ Isomorphism	$\simeq$ Equivalence
$gf = g \circ f = f; g$ if $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$: morphism composition	$\mathrm{id}\colon X \longrightarrow X$ or $1\colon X \longrightarrow X$ Identity morphism
$\mathcal{P}$, $\mathcal{P}_{\mathrm{fin}}$, $\mathcal{P}^+$ Powerset, finite subsets, non-empty subsets	$[\![n, m]\!]$ Set $\{k \mid n \leq k \leq m\}$
$\kappa_i \quad (i \in \mathbb{N})$ Coprojections (if a coproduct is involved)	$\pi_i \quad (i \in \mathbb{N})$ Projections (if a product is involved)
$[f, g]\colon A + B \longrightarrow X$ copairing of $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow X$	$f + g\colon A + B \longrightarrow X + Y$ coproduct $[\kappa_1 f, \kappa_2 g]$ of $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow Y$
$\mathbb{0}$, $\mathbb{1}$ or 0, 1 initial and terminal (when not unit of the tensor in linear logic) objects	$!\colon \mathbb{0} \longrightarrow X$, $!\colon X \longrightarrow \mathbb{1}$ initial/terminal morphism (when not the exponential modality in linear logic)
$F \underset{\eta}{\overset{\epsilon}{\dashv}} G$ F left adjoint to G, with unit η and counit ϵ	
$\mathcal{K}\ell(\mathbb{T})$ or $\mathscr{C}_{\mathbb{T}}$ Kleisli category of the monad $\mathbb{T}\colon \mathscr{C} \longrightarrow \mathscr{C}$	$\mathcal{EM}(\mathbb{T})$ or $\mathscr{C}^{\mathbb{T}}$ Eilenberg-Moore category of $\mathbb{T}\colon \mathscr{C} \longrightarrow \mathscr{C}$
$A^{\vee}$ Join closure of A	

For notational convenience, one may drop the subscripts in natural transformations when the context makes it unambiguous.

A.0.2 Abbreviations

General	**iff**: if and only if **resp.**: respectively **cf.**: see
Categories	**SMC**: symmetric monoidal closed
Linear Logic	**ILL/CLL**: intuitionistic/classical linear logic **MLL**: multiplicative linear logic **MALL**: multiplicative additive linear logic **MELL**: multiplicative exponential linear logic

B. Orthogonal construction

Proposition B.1 Let $\mathbb{C}$ be a small **distributive category**, *i.e.* a small category having finite products $\times$ and coproducts $+$ such that for every $A, B, C \in \mathbb{C}$, the canonical morphism $A \times B + A \times C \longrightarrow A \times (B + C)$ is invertible. Then $\mathrm{FProd}\left(\mathbb{C}^{\mathrm{op}}, \mathbf{Set}\right)$ is an exponential ideal of $\widehat{\mathbb{C}}$.

Proof
Let $P \in \mathrm{FProd}\left(\mathbb{C}^{\mathrm{op}}, \mathbf{Set}\right)$, $Q \in \widehat{\mathbb{C}}$. Showing that $P^Q \in \mathrm{FProd}\left(\mathbb{C}^{\mathrm{op}}, \mathbf{Set}\right)$ amounts to showing that

$$\forall A, B \in \mathbb{C}, \quad P^Q(A+B) \cong P^Q(A) \times P^Q(B)$$

Recall that every presheaf Q is a canonical colimit of representables:

$$Q \;\cong\; \operatorname{colim}\left(\mathbf{y}_{\mathbb{C}} \downarrow Q \xrightarrow{\mathscr{U}} \mathbb{C} \xrightarrow{\mathbf{y}_{\mathbb{C}}} \widehat{\mathbb{C}}\right) \overset{\text{denoted}}{:=} \operatorname*{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathbf{y}_{\mathbb{C}}(X)$$

Thus, we have the natural isomorphisms:

$$\begin{aligned}
P^Q(A+B) &\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(A+B), P^Q\right) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(Q \times \mathbf{y}_{\mathbb{C}}(A+B), P\right) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\left(\operatorname{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathbf{y}_{\mathbb{C}}(X)\right) \times \mathbf{y}_{\mathbb{C}}(A+B), P\right) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\operatorname{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q}\left(\mathbf{y}_{\mathbb{C}}(X) \times \mathbf{y}_{\mathbb{C}}(A+B)\right), P\right) && \text{as } (-) \times \mathbf{y}_{\mathbb{C}}(A+B) \text{ is a left-adjoint} \\
&\cong \lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X) \times \mathbf{y}_{\mathbb{C}}(A+B), P\right) \\
&\cong \lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X \times (A+B)), P\right) \\
&\cong \lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X \times A + X \times B), P\right) && \text{by distributivity} \\
&\cong \lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} P(X \times A + X \times B) \\
&\cong \lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \Big(P(X \times A) \times P(X \times B)\Big) && \text{as } P \in \mathrm{FProd}\left(\mathbb{C}^{\mathrm{op}}, \mathbf{Set}\right) \\
&\cong \Big(\lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} P(X \times A)\Big) \times \Big(\lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} P(X \times B)\Big) \\
&\cong \Big(\lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X \times A), P\right)\Big) \times \Big(\lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X \times B), P\right)\Big) \\
&\cong \Big(\lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X) \times \mathbf{y}_{\mathbb{C}}(A), P\right)\Big) \times \Big(\lim_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(X) \times \mathbf{y}_{\mathbb{C}}(B), P\right)\Big) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\operatorname{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q}\left(\mathbf{y}_{\mathbb{C}}(X) \times \mathbf{y}_{\mathbb{C}}(A)\right), P\right) \times \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\operatorname{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q}\left(\mathbf{y}_{\mathbb{C}}(X) \times \mathbf{y}_{\mathbb{C}}(B)\right), P\right) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\left(\operatorname{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathbf{y}_{\mathbb{C}}(X)\right) \times \mathbf{y}_{\mathbb{C}}(A), P\right) \times \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\left(\operatorname{colim}_{\mathbf{y}_{\mathbb{C}}(X) \longrightarrow Q} \mathbf{y}_{\mathbb{C}}(X)\right) \times \mathbf{y}_{\mathbb{C}}(B), P\right) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(Q \times \mathbf{y}_{\mathbb{C}}(A), P\right) \times \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(Q \times \mathbf{y}_{\mathbb{C}}(B), P\right) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(A), P^Q\right) \times \mathrm{Hom}_{\widehat{\mathbb{C}}}\left(\mathbf{y}_{\mathbb{C}}(B), P^Q\right) \\
&\cong P^Q(A) \times P^Q(B)
\end{aligned}$$

$\blacksquare$

C. Ideal monad

Proposition C.1 The ideal monad

$$\mathcal{I}\colon \begin{cases} \boldsymbol{\mathcal{MDL}at} \longrightarrow \boldsymbol{\mathcal{MDL}at} \\ \quad D \qquad\longmapsto \quad \{\delta\subseteq D \mid \delta \text{ non-empty directed down-set}\} \\ D\xrightarrow{f} D' \longmapsto \quad \mathcal{I}(f) := \begin{cases}\mathcal{I}(D) \longrightarrow \mathcal{I}(D') \\ \delta \qquad\longmapsto \{d'\in D' \mid \exists\, d\in\delta;\ d'\le f(d)\}\end{cases}\end{cases}$$

is well-defined.

Proof

One easily checks that $\mathcal{I}(f)$ preserves $\cap$, and that $\mathcal{I}(f)(\delta)$ is down-closed and closed under $\vee$:

$$\begin{cases} d_1' \le f(d_1) \le f(d_1\vee d_2) \\ d_2' \le f(d_2) \le f(d_1\vee d_2)\end{cases} \Longrightarrow d_1'\vee d_2' \le f(\underbrace{d_1\vee d_2}_{\in\delta})$$

Moreover, we do have a monad:

$$\mathcal{I}(D)\ni\delta \xmapsto{\mathcal{I}(\eta_D)} \{\delta'\in\mathcal{I}(D) \mid \exists\, d\in\delta;\ \delta'\subseteq{\downarrow}d\} \xmapsto{\mu_D} \{d'\in D \mid \exists\, d\in\delta;\ \underbrace{{\downarrow}d'\subseteq{\downarrow}d}_{\Leftrightarrow d'\le d}\} \overset{\subseteq \text{ by down-closedness of } \delta}{=} \delta$$

$$\mathcal{I}(D)\ni\delta \xmapsto{\eta_{\mathcal{I}D}} {\downarrow}\delta \xmapsto{\mu_D} \{d\in D \mid \overbrace{{\downarrow}d\in{\downarrow}\delta}^{\Leftrightarrow {\downarrow}d\subseteq\delta}\} \underset{\supseteq \text{ by down-closedness of } \delta}{=} \delta$$

$$\begin{tikzcd}
\mathcal{I}^3(D)\ni\hat\Phi \arrow[r, mapsto, "\mu_{\mathcal{I}D}"] \arrow[d, mapsto, "\mathcal{I}\mu_D"'] & \{\delta'\in\mathcal{I}D \mid {\downarrow}\delta'\in\hat\Phi\} \arrow[r, mapsto, "\mu_D"] & \{d\in D \mid {\downarrow}{\downarrow}d\in\hat\Phi\} \arrow[d, equal] \\
\{\delta'\in\mathcal{I}(D) \mid \exists\,\Phi\in\hat\Phi;\ \underbrace{\delta'\subseteq\mu\Phi}_{\Leftrightarrow\left[\forall d,\, d\in\delta'\Rightarrow{\downarrow}d\in\Phi\right]}\} \arrow[rr, mapsto, "\mu_D"'] & & \{d\in D \mid \exists\,\Phi\in\hat\Phi;\ \forall d'\in D, \underbrace{d'\in{\downarrow}d}_{\Leftrightarrow d'\le d}\Rightarrow{\downarrow}d'\in\Phi\}
\end{tikzcd}$$

With respect to the equality on the right side: the inclusion from top to bottom is clear. The reverse one stems from the fact that if d is an element of the bottom set, it comes that ${\downarrow}{\downarrow}d\subseteq\Phi\in\hat\Phi$. Indeed: for all $\delta\subseteq{\downarrow}d$, as ${\downarrow}d\in\Phi$ by hypothesis, $\delta\in\Phi$ by down-closedness of Φ. As a result, ${\downarrow}{\downarrow}d\in\hat\Phi$ due to $\hat\Phi$ being down-closed.

∎

Proposition C.2 The category $\boldsymbol{\mathcal{IDL}at}$ can be equivalently described as that with bounded distributive lattices as objects and morphisms given by distributors $f\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$: that is, monotone functions $f\colon Y^\circ\times X\longrightarrow\Sigma$ such that, for all $x\in X$ and $y\in Y$, $f(-,x) : Y^\circ\longrightarrow\Sigma$ and $f(y,-) : X\longrightarrow\Sigma$ preserve finite meets, with identities given by $\mathrm{id}(x',x) = [\,x'\le x\,]$ and composition $f\circ g\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ of $g : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $f : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ given by

$$(f\circ g)(z,x) = \bigvee_{Z_0\subseteq_{\mathrm{fin}}Z} \left[z\le\bigvee Z_0\right]\wedge\left[\ \bigwedge_{z_0\in Z_0}\ \bigvee_{y\in Y} f(z_0,y)\wedge g(y,x)\ \right] .$$

Proof

We cannot rely on the fact that $\boldsymbol{\mathcal{MDL}at}$ is closed for the tensor product $\otimes$, as this is not known (cf [Bara; Barb]). But Σ does turn out to be a exponentiating object in $\boldsymbol{\mathcal{MDL}at}$:

- We can readily check it by hand, by showing that maps $A \longrightarrow \mathcal{I}(B) \in \mathcal{MDLat}$ are in one-to-one correspondence (naturally in A, B) with bihomomorphisms $B^\circ \times A \longrightarrow \Sigma$.
- Or, better: by compact closedness of $\mathcal{IDLat}$ (Theorem 3.2):

$$\mathrm{Hom}_{\mathcal{IDLat}}\,(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{IDLat}}\,(A, B^\circ \otimes C)$$

naturally in A, B, C. So we have the following natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{MDLat}}\,(B^\circ \otimes A, \Sigma) &\cong \mathrm{Hom}_{\mathcal{MDLat}}\,(A \otimes B^\circ, \Sigma) \\
&\cong \mathrm{Hom}_{\mathcal{MDLat}}\,(A \otimes B^\circ, \mathcal{I}(\Sigma)) && \text{as } \Sigma \cong \mathcal{I}(\Sigma) \\
&\cong \mathrm{Hom}_{\mathcal{IDLat}}\,(A \otimes B^\circ, \Sigma) \\
&\cong \mathrm{Hom}_{\mathcal{IDLat}}\,(A, B \otimes \Sigma) \\
&\cong \mathrm{Hom}_{\mathcal{IDLat}}\,(A, B) \\
&\cong \mathrm{Hom}_{\mathcal{MDLat}}\,(A, \mathcal{I}(B))
\end{aligned}$$

■

Proposition C.3 $\mathcal{I}$ is a strong monad.

Proof
Let us show that we have a strength $t_{A,B} \colon A \otimes \mathcal{I}(B) \longrightarrow \mathcal{I}(A \otimes B)$, for every $A, B \in \mathcal{MDLat}$. By universal property of the tensor product, giving such a map amounts to giving a bihomomorphism $\tilde{t}_{A,B} \colon A \times \mathcal{I}(B) \longrightarrow \mathcal{I}(A \otimes B)$. We set

$$\tilde{t}_{A,B} \colon \begin{cases} A \times \mathcal{I}(B) & \longrightarrow \mathcal{I}(A \otimes B) \\ a,\ \emptyset \neq \beta \subseteq B & \longmapsto \downarrow \{a \otimes b\}_{b \in \beta} \end{cases}$$

- $\tilde{t}_{A,B}$ is well-defined, *i.e.* $\downarrow \{a \otimes b\}_{b \in \beta}$ is an ideal of $A \otimes B$: we show that $\downarrow \{a \otimes b\}_{b \in \beta}$ is the smallest ideal $\left\langle \{a \otimes b\}_{b \in \beta} \right\rangle$ containing $\{a \otimes b\}_{b \in \beta}$
  - $\left\langle \{a \otimes b\}_{b \in \beta} \right\rangle \subseteq \downarrow \{a \otimes b\}_{b \in \beta}$: Let $d \leq \bigvee_i a \otimes b_i$. Since $\bigvee_i a \otimes b_i = a \otimes \bigvee_i b_i$ by [Fra76, Theorem 2.6], and $\bigvee_i b_i \in \beta$ as β is directed, it comes that d is lower than an element of $\{a \otimes b\}_{b \in \beta}$.
  - the other inclusion is clear.
- $\tilde{t}_{A,B}$ *is a bihomomorphism*:
  - for every $\beta \in \mathcal{I}(B)$, $\tilde{t}_{A,B}(-, \beta)$ is a homomorphism: for every $a, a' \in A$,

$$\begin{aligned}
\tilde{t}_{A,B}(a \wedge a', \beta) &:= \downarrow \{(a \otimes b) \wedge (a' \otimes b)\}_{b \in \beta} \\
&= \downarrow \{a \otimes b\}_{b \in \beta} \cap \downarrow \{a' \otimes b\}_{b \in \beta} \qquad (\circledast) \\
&= \tilde{t}_{A,B}(a, \beta) \cap \tilde{t}_{A,B}(a', \beta)
\end{aligned}$$

  About the equality ($\circledast$): inclusion $\subseteq$ is clear. As for $\supseteq$: with obvious notations, if $c \leq a \otimes b$ and $c \leq a' \otimes b'$, then $c \leq a \otimes b \leq a \otimes (b \vee b')$ and $c \leq a \otimes b \leq a' \otimes (b \vee b')$, and since $b \vee b' \in \beta$, the result follows.
  - for every $a \in A$, $\tilde{t}_{A,B}(a, -)$ is a homomorphism: for every $\beta, \beta' \in \mathcal{I}(B)$,

$$\begin{aligned}
\tilde{t}_{A,B}(a, \beta \cap \beta') &:= \downarrow \{a \otimes b\}_{b \in \beta \cap \beta'} \\
&= \downarrow \{a \otimes b\}_{b \in \beta} \cap \downarrow \{a \otimes b'\}_{b' \in \beta'} \qquad (\circledcirc) \\
&= \tilde{t}_{A,B}(a, \beta) \cap \tilde{t}_{A,B}(a, \beta')
\end{aligned}$$

  Again, $\subseteq$ is obvious in the equality ($\circledcirc$). As for $\supseteq$: if $c \leq a \otimes b$ and $c \leq a \otimes b'$, then $c \leq (a \otimes b) \wedge (a \otimes b') = a \otimes (b \wedge b')$ and since $b \wedge b' \in \beta \cap \beta'$, the result follows.

Therefore, there exists an unnatural transformation:

$$t_{A,B}\colon \begin{cases} A \otimes \mathcal{I}(B) & \longrightarrow \mathcal{I}(A \otimes B) \\ \bigwedge_i a_i \otimes \beta_i & \longmapsto \bigcap_i \downarrow \{a_i \otimes b\}_{b \in \beta_i} \end{cases}$$

It turns out to be natural in A, B, as shown by diagram chasing, for all $f\colon A \longrightarrow A'$, $g\colon B \longrightarrow B'$:

$$\begin{tikzcd}
\bigwedge_i a_i \otimes \beta_i \arrow[r, mapsto, "f \otimes \mathcal{I}g"] \arrow[d, mapsto, "t_{A,B}"'] & \bigwedge_i f(a_i) \otimes \{b' \in B \mid \exists\, b \in \beta_i;\ b' \leq g(b)\} \arrow[r, mapsto, "t_{A',B'}"] & \bigcap_i \downarrow \{f(a_i) \otimes b' \mid \exists\, b \in \beta_i;\ b' \leq g(b)\} \arrow[d, equal] \\
\bigcap_i \downarrow \{a_i \otimes b_i\}_{b_i \in \beta_i} \arrow[rr, mapsto, "\mathcal{I}(f \otimes g)"'] & & \{d' \in A' \otimes B' \mid \exists\, d \leq \bigwedge_i a_i \otimes \underbrace{b_i}_{\in \beta_i};\ d' \leq (f \otimes g)(d)\}
\end{tikzcd}$$

The equality stemming from f, g and $f \otimes g$ being monotone and preserving $\wedge$. We now check that this natural transformation $t_{A,B}$ is a strength.

$$\begin{tikzcd}
\top \otimes \alpha \arrow[r, mapsto, "t_{\Sigma,A}"] \arrow[dr, mapsto, "\lambda_{\mathcal{I}A}"'] & \downarrow \{\top \otimes a\}_{a \in \alpha} = \Sigma \otimes A \arrow[d, mapsto, "\mathcal{I}(\lambda_A)"] \\
& \{a' \in A \mid \exists\, a'' \leq \top \otimes \underbrace{a}_{\in \alpha};\ a' \leq \lambda_A a''\} = A
\end{tikzcd}
\qquad
\begin{tikzcd}
\bot \otimes \alpha \arrow[r, mapsto, "t_{\Sigma,A}"] \arrow[dr, mapsto, "\lambda_{\mathcal{I}A}"'] & \downarrow \{\bot \otimes a\}_{a \in \alpha} = \{\bot \otimes a\}_{a \in \alpha} \arrow[d, mapsto, "\mathcal{I}(\lambda_A)"] \\
& \{a' \in A \mid \exists\, a'' \leq \bot \otimes \underbrace{a}_{\in \alpha};\ a' \leq \lambda_A a''\} = \downarrow \alpha = \alpha
\end{tikzcd}$$

$$\begin{tikzcd}
(a \otimes b) \otimes \gamma \arrow[r, mapsto, "t_{A \otimes B,C}"] \arrow[d, mapsto, "\alpha_{A,B,\mathcal{I}C}"'] & \downarrow \{(a \otimes b) \otimes c\}_{c \in \gamma} \arrow[r, mapsto, "\mathcal{I}(\alpha_{A,B,C})"] & \{d' \in (A \otimes B) \otimes C \mid \exists\, d \leq (a \otimes b) \otimes \underbrace{c}_{\in \gamma};\ d' \leq \alpha_{A,B,C}\, d\} \arrow[d, equal] \\
a \otimes (b \otimes \gamma) \arrow[r, mapsto, "1 \otimes t_{B,C}"'] & a \otimes \downarrow \{b \otimes c\}_{c \in \gamma} \arrow[r, mapsto, "t_{A,B \otimes C}"'] & \{d' \in (A \otimes B) \otimes C \mid \exists\, c \in \gamma,\ e \leq b \otimes c;\ d' \leq a \otimes e\}
\end{tikzcd}$$

$$\begin{tikzcd}
a \otimes b \arrow[r, mapsto, "\mathrm{id} \otimes \eta_B"] \arrow[dr, mapsto, "\eta_{A \otimes B}"'] & a \otimes \downarrow b \arrow[d, mapsto, "t_{A,B}"] \\
& \downarrow \{a \otimes b'\}_{b' \leq b} = \downarrow(a \otimes b)
\end{tikzcd}$$

$$\begin{tikzcd}
a \otimes \Phi \arrow[r, mapsto, "t_{A,\mathcal{I}B}"] \arrow[d, mapsto, "1 \otimes \mu_B"'] & \downarrow \{a \otimes \beta\}_{\beta \in \Phi} \arrow[r, mapsto, "\mathcal{I}(t_{A,B})"] & \{\delta \in \mathcal{I}(A \otimes B) \mid \exists\, \theta \leq a \otimes \underbrace{\beta}_{\in \Phi};\ \delta \subseteq t_{A,B}\, \theta\} \arrow[d, mapsto, "\mu_{A \otimes B}"] \\
a \otimes \{b \in B \mid \downarrow b \in \Phi\} \arrow[r, mapsto, "t_{A,B}"'] & \downarrow \{a \otimes b\}_{\downarrow b \in \Phi} \arrow[r, equal] & \{c \in A \otimes B \mid \exists\, \theta \leq a \otimes \underbrace{\beta}_{\in \Phi};\ \downarrow c \subseteq t_{A,B}\, \theta\}
\end{tikzcd}$$

The last equality deserves an explanation:

- $\subseteq$: if $c \leq a \otimes b$ where $\downarrow b \in \Phi$, then setting $\theta := a \otimes \downarrow b$ yields the result, since $\downarrow c \subseteq t_{A,B}\theta = \downarrow \{a \otimes b'\}_{b' \leq b} = \downarrow(a \otimes b)$ is equivalent to $c \leq a \otimes b$.
- $\supseteq$: if $\downarrow c \subseteq t_{A,B}\, \theta$ where $\theta \leq a \otimes \beta$ for some $\beta \in \Phi$, then $\downarrow c \subseteq t_{A,B}(a \otimes \beta) = \downarrow \{a \otimes b\}_{b \in \beta}$, which implies that $c \leq a \otimes b$ for some $b \in \beta$, thus satisfying $\downarrow b \subseteq \beta \in \Phi$ hence $\downarrow b \in \Phi$.

$\blacksquare$

Lemma C.1 $\mathcal{I}$ is a commutative monad.

Proof
By Proposition C.3, it is strong for the strength:

$$t_{A,B}\colon \begin{cases} A\otimes \mathcal{I}(B) & \longrightarrow \mathcal{I}(A\otimes B) \\ \bigwedge_i a_i \otimes \beta_i & \longmapsto \bigcap_i {\downarrow}\left\{a_i\otimes b\right\}_{b\in\beta_i} \end{cases}$$

Let us show that the costrength

$$t'_{A,B} := \mathcal{I}A\otimes B \xrightarrow{\sigma_{A,B}} B\otimes \mathcal{I}A \xrightarrow{t_{B,A}} \mathcal{I}(B\otimes A) \xrightarrow{\mathcal{I}(\sigma_{B,A})} \mathcal{I}(A\otimes B)$$

is given by

$$t'_{A,B}\colon \begin{cases} \mathcal{I}(A)\otimes B & \longrightarrow \mathcal{I}(A\otimes B) \\ \bigwedge_i \alpha_i \otimes b_i & \longmapsto \bigcap_i {\downarrow}\left\{a\otimes b_i\right\}_{a\in\alpha_i} \end{cases}$$

It is enough to make sure that they coincide on basis elements, the equality will follow by preservation of $\wedge$:

$$\alpha\otimes b \xmapsto{\sigma_{A,B}} b\otimes\alpha \xmapsto{t_{B,A}} {\downarrow}\left\{b\otimes a\right\}_{a\in\alpha} \xmapsto{\mathcal{I}(\sigma_{B,A})} \{c\in A\otimes B \mid \exists\, d\leq b\otimes \overbrace{a}^{\in\alpha};\ c\leq \sigma_{B,A}d\} = {\downarrow}\left\{a\otimes b\right\}_{a\in\alpha}$$

This is enables us to prove that the relevant square commutes without much effort:

$$\begin{tikzcd}
\alpha\otimes\beta \arrow[r, mapsto, "t_{\mathcal{I}A,B}"] \arrow[d, mapsto, "t'_{A,\mathcal{I}B}"'] & {\downarrow}\left\{\alpha\otimes b\right\}_{b\in\beta} \arrow[r, mapsto, "\mathcal{I}(t'_{A,B})"] & \{\delta \mid \exists\,\theta\leq\alpha\otimes\overbrace{b}^{\in\beta};\ \delta\leq t'_{A,B}\theta\} \arrow[r, mapsto, "\mu_{A\otimes B}"] & \{d \mid \exists\,\theta\leq\alpha\otimes\overbrace{b}^{\in\beta};\ {\downarrow}d\leq t'_{A,B}\theta\} \arrow[d, equal] \\
{\downarrow}\left\{a\otimes\beta\right\}_{a\in\alpha} \arrow[rr, mapsto, "\mathcal{I}(t_{A,B})"'] & & \{\delta \mid \exists\,\theta\leq\underbrace{a}_{\in\alpha}\otimes\beta;\ \delta\leq t_{A,B}\theta\} \arrow[r, mapsto, "\mu_{A\otimes B}"'] & \{d \mid \exists\,\theta\leq\underbrace{a}_{\in\alpha}\otimes\beta;\ {\downarrow}d\leq t_{A,B}\theta\}
\end{tikzcd}$$

The set equality boils down to the equivalence, for all $d\in A\otimes B$:

$$\exists\,\theta'\leq\alpha\otimes\underbrace{b}_{\in\beta};\ {\downarrow}d\leq t'_{A,B}\theta' \quad\Longleftrightarrow\quad \exists\,\theta\leq\underbrace{a}_{\in\alpha}\otimes\beta;\ {\downarrow}d\leq t_{A,B}\theta$$

which holds:

- $\Longrightarrow$: As

$$\downarrow d\subseteq t'_{A,B}(\alpha\otimes b) = \mathcal{I}(\sigma_{B,A})\ \overbrace{t_{B,A}(b\otimes\alpha)}^{={\downarrow}\left\{b\otimes a\right\}_{a\in\alpha}} = \{d'\in A\otimes B \mid \exists\, c\leq b\otimes\overbrace{a}^{\in\alpha};\ d'\leq\sigma_{B,A}c\}$$

  then there exists $c\leq b\otimes a$ for some $a\in\alpha$ such that

$$d\leq\sigma_{B,A}c\leq\sigma_{B,A}(b\otimes a) = a\otimes b \text{ where } a\in\alpha,\ b\in\beta$$

  So by putting $\theta := a\otimes{\downarrow}b\leq a\otimes\beta$, we have

$${\downarrow}d\leq{\downarrow}(a\otimes b) = {\downarrow}\left\{a\ \otimes\ b'\right\}_{b'\leq b} = t_{A,B}(\theta)$$

- $\Longleftarrow$: Suppose that there exists $\theta\leq a\otimes\beta$ for some $a\in\alpha$ such that

$${\downarrow}d\subseteq t_{A,B}\,\theta\subseteq t_{A,B}(a\otimes\beta) = {\downarrow}\left\{a\ \otimes\ b\right\}_{b\in\beta}$$

  It comes that $d\leq a\otimes b$ for some $a\in\alpha$, $b\in\beta$. By setting $\theta' := {\downarrow}a\otimes b\leq\alpha\otimes b$, we get

$${\downarrow}d\subseteq{\downarrow}(a\otimes b) = {\downarrow}\left\{a'\ \otimes\ b\right\}_{a'\leq a} = t'(\theta')$$

$\blacksquare$

D. Model of Linear Logic

Proposition D.1 $\boldsymbol{\mathcal{MDL}at}$ has biproducts.

Proof
$\boldsymbol{\mathcal{MSL}at}$ is the category of models of an equational theory (idempotent commutative monoids), and as such, has products and coproducts, that happen to coincide in $\boldsymbol{\mathcal{MSL}at}$ (biproducts).

For all $A_1, A_2 \in \boldsymbol{\mathcal{MSL}at}$, $A_1 + A_2$ is the free algebra (in the universal algebraic sense) generated by A_1 and A_2, comprised of all the terms built up from elements of $A_1 \cup A_2$, quotiented by the equational identities. It is given by:

$$
\begin{aligned}
A_1 + A_2 \quad & := \{a_1 \wedge a_2 \mid a_1 \in A_1,\, a_2 \in A_2\} \\
\kappa_1 \colon A_1 \longrightarrow A_1 + A_2 & := a_1 \longmapsto a_1 \wedge \top_2 \\
\kappa_2 \colon A_2 \longrightarrow A_1 + A_2 & := a_2 \longmapsto \top_1 \wedge a_2
\end{aligned}
$$

where the top element is $\top_1 \wedge \top_2$, and meets are taken component-wise. Furthermore, if A_1, A_2 are distributive, $A_1 + A_2$ can endowed with a distributive structure by setting:

$$
(a_1 \wedge a_2) \vee (a_1' \wedge a_2') := (a_1 \vee a_1') \wedge (a_2 \vee a_2') \in A_1 + A_2 \tag{D.1}
$$

As it happens, the distributive law does indeed hold in $A_1 + A_2$ (we only need to check one of the two binary distributivity laws, as the other one follows [Bir40, Theorem I.6.9]):

$$
\begin{aligned}
(a_1'' \wedge a_2'') \wedge \Big((a_1 \wedge a_2) \vee (a_1' \wedge a_2')\Big) &= (a_1'' \wedge a_2'') \wedge \Big((a_1 \vee a_1') \wedge (a_2 \vee a_2')\Big) && \text{by eq. (D.1)} \\
&= \big(a_1'' \wedge (a_1 \vee a_1')\big) \wedge \big(a_2'' \wedge (a_2 \vee a_2')\big) \\
&= \big((a_1'' \wedge a_1) \vee (a_1'' \wedge a_1')\big) \wedge \big((a_2'' \wedge a_2) \vee (a_2'' \wedge a_2')\big) && \text{by distributivity of } A_1, A_2 \\
&= \big((a_1'' \wedge a_1) \wedge (a_2'' \wedge a_2)\big) \vee \big((a_1'' \wedge a_1') \wedge (a_2'' \wedge a_2')\big) && \text{by eq. (D.1)} \\
&= \big((a_1'' \wedge a_2'') \wedge (a_1 \wedge a_2)\big) \vee \big((a_1'' \wedge a_2'') \wedge (a_1' \wedge a_2')\big)
\end{aligned}
$$

It comes that when $A_1, A_2 \in \boldsymbol{\mathcal{MDL}at}$, the $\boldsymbol{\mathcal{MSL}at}$ coproduct $A_1 + A_2$ remains in $\boldsymbol{\mathcal{MDL}at}$. Therefore, as $\boldsymbol{\mathcal{MDL}at}$ fully embeds in $\boldsymbol{\mathcal{MSL}at}$, $\boldsymbol{\mathcal{MDL}at}$ has coproducts, and we can show in a similar way that it has biproducts. ■

Lemma D.1 — Coequalisers of identity and permutations of n-th tensor powers.
Let $A \in \boldsymbol{\mathcal{MDL}at}$, $n \in \mathbb{N}$, and $\sigma_1, \ldots, \sigma_k \colon A^{\otimes n} \longrightarrow A^{\otimes n} \in \boldsymbol{\mathcal{MDL}at}$ be $1 \leq k \leq n!$ permutations of the n-th tensor power $A^{\otimes n}$. Then the coequaliser of the parallel morphisms $\sigma_1, \ldots, \sigma_k \colon A^{\otimes n} \longrightarrow A^{\otimes n}$ exists in $\boldsymbol{\mathcal{MDL}at}$.

Proof
Let $\sigma' \colon A^{\otimes n} \longrightarrow A^{\otimes n}$ be a permutation of the n-th tensor power $A^{\otimes n}$. It is defined as:

$$
\sigma' \colon \begin{cases} A^{\otimes n} & \longrightarrow A^{\otimes n} \\ \bigwedge_i a_1^i \otimes \cdots \otimes a_n^i & \longmapsto \bigwedge_i a_{\sigma'(1)}^i \otimes \cdots \otimes a_{\sigma'(1)}^i \end{cases}
$$

One checks that σ' preserves joins:

$$
\begin{aligned}
\sigma'\Big(\bigwedge_i a_1^i\otimes\cdots\otimes a_n^i \;\vee\; \bigwedge_j b_1^j\otimes\cdots\otimes b_n^j\Big) &= \sigma'\Big(\bigwedge_{i,j}\big(a_1^i\vee b_1^j\big)\otimes\cdots\otimes\big(a_n^i\vee b_n^j\big)\Big)\\
&= \bigwedge_{i,j}\big(a_{\sigma'(1)}^i\vee b_{\sigma'(1)}^j\big)\otimes\cdots\otimes\big(a_{\sigma'(n)}^i\vee b_{\sigma'(n)}^j\big)\\
&= \bigwedge_i a_{\sigma'(1)}^i\otimes\cdots\otimes a_{\sigma'(n)}^i \;\vee\; \bigwedge_j b_{\sigma'(1)}^j\otimes\cdots\otimes b_{\sigma'(n)}^j\\
&= \sigma'\Big(\bigwedge_i a_1^i\otimes\cdots\otimes a_n^i\Big)\vee\sigma'\Big(\bigwedge_j b_1^j\otimes\cdots\otimes b_n^j\Big)
\end{aligned}
$$

So $\sigma'\colon A^{\otimes n}\longrightarrow A^{\otimes n}$ can be seen as a morphism in the category $\boldsymbol{\mathcal{DL}at}$ of distributive lattices and lattice (join- and meet-preserving) morphisms. As $\boldsymbol{\mathcal{DL}at}$ is a category of models of an algebraic theory, it is cocomplete (see [ARV11; nLaa]), and the coequaliser $A^{\otimes n}/\mathfrak{S}'_n$ of the parallel morphisms $\mathrm{id}_{A^{\otimes n}},\,\sigma_1,\dots,\sigma_k\colon A^{\otimes n}\longrightarrow A^{\otimes n}$ exists in $\boldsymbol{\mathcal{DL}at}$:

$$
\begin{tikzcd}
A^{\otimes n} \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_k"', "\vdots"] & A^{\otimes n} \arrow[r, "\varphi"] & A^{\otimes n}/\mathfrak{S}'_n
\end{tikzcd}
$$

Let us now show that $A^{\otimes n}/\mathfrak{S}'_n$ satisfies the coequaliser universal property in $\boldsymbol{\mathcal{MDL}at}$, that is:

$$
\begin{tikzcd}
A^{\otimes n} \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_k"', "\vdots"] & A^{\otimes n} \arrow[r, "\varphi"] \arrow[dr, "\forall f"'] & A^{\otimes n}/\mathfrak{S}'_n \arrow[d, dotted, "\exists!\,h"] \\
 & & B
\end{tikzcd}
$$

Let $f\colon A^{\otimes n}\longrightarrow B$ correspond to a cocone as depicted above, and let us construct a unique cocone morphism $h\colon A^{\otimes n}/\mathfrak{S}'_n\longrightarrow B$. By universal property of the tensor (up to rebracketing, but this is taken care of by Mac Lane's coherence theorem for monoidal categories [Lan78, Chapter 7]), precomposing f by the canonical map $\pi\colon A^n\longrightarrow A^{\otimes n}$ yields a multimorphism $f'\colon A^n\longrightarrow A^{\otimes n}$:

$$
\begin{tikzcd}
A^{\otimes n} \arrow[r, "f"] & A^{\otimes n}/\mathfrak{S}'_n \\
A^n \arrow[u, "\pi"] \arrow[ur, "f'"'] &
\end{tikzcd}
$$

Therefore, if a suitable $h\colon A^{\otimes n}/\mathfrak{S}'_n\longrightarrow B$ existed, it would be meet-preserving and since $h\varphi\,\sigma_1 = f\,\sigma_1$, hence $h\varphi = f$ and $h\varphi\pi = f\pi = f'$, it would necessarily be uniquely determined as follows:

$$
h\colon\;\begin{cases}
A^{\otimes n}/\mathfrak{S}'_n & \longrightarrow\; B\\
\underbrace{\varphi\Big(\bigwedge_i a_1^i\otimes\cdots\otimes a_n^i\Big)}_{=\bigwedge_i \varphi(a_1^i\otimes\cdots\otimes a_n^i)} & \longmapsto\; \bigwedge_i \underbrace{h\Big(\varphi\big(a_1^i\otimes\cdots\otimes a_n^i\big)\Big)}_{=h\,\varphi\,\pi\left((a_1^i,\cdots,a_n^i)\right)=f'\left((a_1^i,\cdots,a_n^i)\right)}
\end{cases}
$$

This expression is also sufficient: let $l\in[\![1,k]\!]$. The permutation $\sigma_l\colon A^{\otimes n}\longrightarrow A^{\otimes n}$ is obtained by universal property of the tensor product for the corresponding permutation $\sigma'_l\colon A^n\longrightarrow A^n$ of the n-th cartesian power A^n postcomposed by the canonical map $\pi\colon A^n\longrightarrow A^{\otimes n}$. As a result:

$$
\begin{aligned}
f'\colon A^n\longrightarrow A^{\otimes n} &= f\pi\\
&= f\sigma_l\pi && \text{as } f = f\sigma_l\\
&= f\pi\sigma'_l && \text{as } \sigma_l\pi = \pi\sigma'_l\\
&= f'\sigma'_l
\end{aligned}
$$

Consequently, such an $h := \bigwedge_i \varphi(a_1^i \otimes \cdots \otimes a_n^i) \longmapsto f'\big((a_1^i, \cdots, a_n^i)\big)$ is well-defined.

■