UNIVERSITY OF CAMBRIDGE

Ideal Distributors

école normale supérieure paris–saclay

From Cartesian to Ideal Distributors

Fiore and Joyal's cartesian closed bicategory of cartesian categories and cartesian distributors:

- Cartesian categories ⇝ Meet-semilattices
- Presheaf construction ⇝ Ideal completion

  (Presheaf construction: Downset completion $\mathcal{D}(-)$ for posets; Ideal completion: Orthogonal subcategory construction)

Categories

$\mathcal{MDLat}$: bounded distributive lattices and meet-preserving maps

$\mathcal{MSLat}$: bounded meet-semilattices and meet-preserving maps

$\mathcal{IDLat}$: Kleisli category of the ideal monad $\mathcal{I}\colon \mathcal{MDLat} \to \mathcal{MDLat}$

Constraints

- Meet-semilattice
  - Orthogonal construction
  - Maximum amount of information contained in tokens
- Distributivity
  - Exponential ideal ⇝ Day's reflection theorem
  - Fraser's theorem ⇝ Restrict $\otimes$ from $\mathcal{MSLat}$ to $\mathcal{MDLat}$

Compact closedness

A symmetric monoidal category $(\mathscr{C}, \otimes, I, \sigma)$ is **compact closed** $\iff$ every $A \in \mathscr{C}$ has an adjoint $A^* \in \mathscr{C}$ (called *dual*) in the bicategorical delooping of $\mathscr{C}$.

Necessary and sufficient conditions on co/units

$\mathcal{IDLat}$ is compact closed with units $i_A\colon \Sigma \multimap A \otimes A^\circ$ and counits $e_A\colon A^\circ \otimes A \multimap \Sigma$ $\iff i_A(\bot) := \downarrow D$ for some $\vee$-closed $D \subseteq A \otimes A^\circ$ such that $\forall a_0 \in A$:

- $\forall \bigwedge_i a'_i \otimes a''_i \in D, \quad \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j \le a_0$
- $\exists \bigwedge_i a'_i \otimes a''_i \in D; \quad \bigwedge_{j;\, e(a''_j \otimes a_0) = \{\bot\}} a'_j = a_0$
- $\forall \bigwedge_i a'_i \otimes a''_i \in D, \quad a_0 \le \bigvee_{j;\, e(a_0 \otimes a'_j) = \{\bot\}} a''_j$
- $\exists \bigwedge_i a'_i \otimes a''_i \in D; \quad a_0 = \bigvee_{j;\, e(a_0 \otimes a'_j) = \{\bot\}} a''_j$

$\mathcal{IDLat}$ is compact closed

The category $\mathcal{IDLat}$ is compact closed, with

- counits $e_A\colon A^\circ \otimes A \multimap \Sigma$:

$$e_A(a' \otimes a) = \downarrow(a' \le a)$$

- units $i_A\colon \Sigma \multimap A \otimes A^\circ$:

$$i_A(\bot) = \downarrow\left(\{\bot \otimes a \wedge a \otimes \top \mid a \in A\}^\vee\right)$$
$$i_A(\top) = A \otimes A^\circ$$

Model of full Linear Logic

$\mathcal{IDLat} := \mathcal{K}\ell(\mathcal{I})$ is a degenerate model of classical linear logic.

Dualisation operation

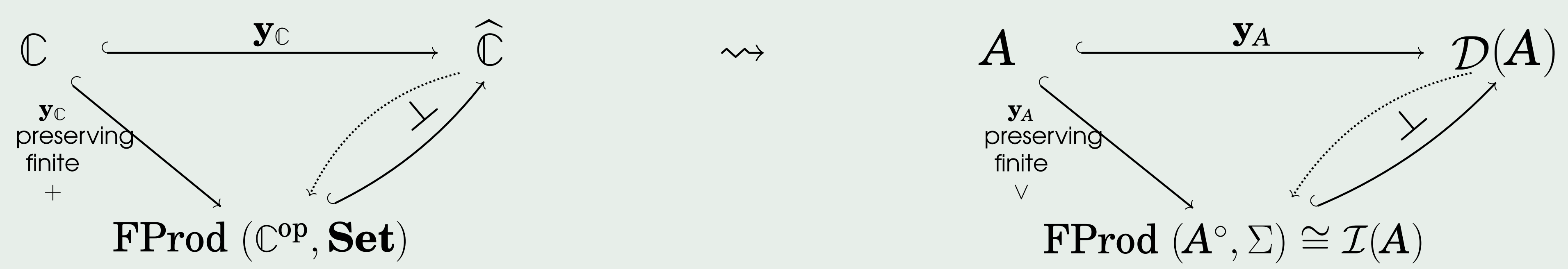
In $\mathcal{IDLat}$, the dualisation operation $(-)^*$ given on every morphism $f\colon A \multimap B$ by

$$\begin{aligned} f^* := \;& \rho^{-1} ; \eta ; \mathcal{I}\big((\eta \otimes i) ; t ; \mathcal{I}(t') ; \mu\big) ; \mu \\ & ; \mathcal{I}\Big(\eta \otimes \big((f \otimes i) ; t ; \mathcal{I}(t') ; \mu\big) ; t ; \mathcal{I}(t') ; \mu\Big) ; \mu \\ & ; \mathcal{I}\big(\alpha^{-1} ; (e \otimes \eta) ; t ; \mathcal{I}(t') ; \mu\big) ; \mu ; \mathcal{I}(\lambda) \end{aligned}$$

simplifies to

$$f^*\colon \begin{cases} B^\circ & \to \mathcal{I}(A^\circ) \\ b & \longmapsto \{a \in A^\circ \mid b \in f(a)\} \end{cases}$$

Orthogonal construction



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