Tricocycloids, Effect Monoids and Effectuses

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M2 Research Internship

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Overview

General context

The discrete distribution monad $\mathcal{D}_{[0,1]}$, associating to every set X the set of finitely supported probability distribution over X, has been shown to be a linear exponential monad by Richard Garner in [Gar18]. Beside modelling the exponential modality ? of linear logic, linear exponential monads are argued, in this article, to be a suitable setting for 'hypernormalisation', a categorically well-behaved generalisation of normalisation of subprobability distributions. But the story does not end here: $\mathcal{D}_{[0,1]}$ being linear exponential raises a couple of compelling questions, all more or less oddly related to quantum mechanics/computation and/or quantum logic – the logic of boolean observables in quantum mechanics –, some of which we have tried to partially address throughout my internship.

Research problem

On the one hand, what makes $\mathcal{D}_{[0,1]}$ a monad is the fact that $[0,1]$ is an effect monoid, *i.e.* a monoid in the category of effect algebras. Effect algebras have been extensively used over the past few years in quantum mechanical foundations, insofar as orthomodular lattices – an abstraction of the lattice of closed subspaces of a Hilbert space, representing experimental propositions about quantum observables – as well as the set of quantum effects – self-adjoint bounded linear operators on a Hilbert space, representing quantum observables – are special cases thereof. They are mathematical structures that can be thought of as generalising both probability and propositions. For an arbitrary effect monoid M, the generalised distribution monad $\mathcal{D}_M$ with coefficients in M remains a monad, but one may wonder if it remains linear exponential as well.

One the other hand, what makes $\mathcal{D}_{[0,1]}$ linear exponential is the fact that $(0,1)$ is a symmetric tricocycloid in the symmetric monoidal category $(\mathbf{Set}, \times)$, as noticed by Garner. Tricocycloids, introduced by Ross Street, are quantum algebraic objects generalising Hopf algebras (of which quantum groups are special cases) and satisfying a cohomological 3-cocycle condition. One may wonder how this condition is related to the problem at hand, and try to generalise the situation to more general instances of tricocycloids.

Last but not least, effectus theory, a new branch of categorical logic developed by Jacobs, Cho and the Westerbaan brothers, comes in handy when it comes to studying the generalised distribution monad. Effectuses model continuous, discrete and quantum logic and probability. Their set of scalars happens to form an effect monoid, and they induce a state-and-effect triangle, which proves to be a valuable point of vue.

Your contribution

We have proved some lemmas relating effect monoids, tricocycloids and effectuses, provided they satisfy some conditions reminiscent of a form of generalised 'normalisation' (of probability subdistributions). Our notions of effect monoids with normalisation and tricycloids with left/double cancellation are directly related to Jacobs' notion of effectus with normalisation, and clarify what happens in the introductory example of $\mathcal{D}_{[0,1]}$. More precisely, given an effectus with normalisation – such as the Kleisli category of a 'probability monad' like the discrete distribution/continuous Giry/Radon/Kantorovitch ones, to name a few –, its effect monoid of scalars has normalisation in our sense, and if we remove the scalars 0 and 1 (corresponding to the two coprojections κ_1, κ_2), it also forms a symmetric tricocycloid with left/double cancellation. On top of that, we have shown that

- the categories of effect monoids with normalisation and of symmetric left/double cancellative tricocycloids in Set are isomorphic
- when the effect monoid M has normalisation, the convex spaces over M can be given by binary sums, the $\mathcal{D}_M$-algebras are idempotent commutative monoids for the tensor induced by the tricocycloid associated to M, and $\mathcal{D}_M$ is linear exponential

- we have exhibited the generalised distribution monad $\mathcal{D}_{M_{\mathbb{B}}}$ over the scalars of an effectus whose objects are finite coproducts of the terminal object 1 as the monad associated to the Lawvere theory $\mathbb{B}^{\mathrm{op}}$, due to $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})$ and $\mathbb{B}$ being isomorphic. In the general case, we have seen that $\mathbb{B}$ can be embedded, under some conditions, in the category of presheaves over $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})^{\mathrm{op}}$, which gives a natural explanation of the 'origin' of the state functor
- we have sketched some ideas to come up with models of linear logic based on the generalised distribution monad $\mathcal{D}_M$

Arguments supporting its validity

We have implemented the proof of the correspondence between effect monoids with normalisation and symmetric left/double cancellative tricocycloids in the platform for deductive program verification Why3. There remains a lot to be done, but hopefully, by bringing the machinery of effectus theory into the picture, we will able to leverage its power in the future to further investigate the matter and bridge the gap between linear logic, quantum logic/computation, and the quantum algebraic theory surrounding tricocycloids.

Summary and future work

There remains so much to be done! Whether it be on the linear logic side (models of LL, comparing with Girard's quantum coherence spaces, etc...), on the effectus theoretic one (how about bringing other effectus-theoretic tools into the picture? And what can we tell about other examples of effectuses than the ones we primarily focused on?), or even on the quantum algebraic one (such as having stronger links with non-abelian cohomology) and the categorical probability one (what about other probability monads? Now that effectus theory has paved the way in this direction, to what extent can we generalise the phenomenon?). But by establishing a link between effect monoids, tricocycloids and effectuses, we now have a track that may prove valuable to understand how all these fruitful notions are related.

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I've had a marvellous stay at Macquarie, so that I have only one desire: coming back as soon as possible!

Prerequisites

We assume familiarity with category theory (categories, functors, natural transformations, adjoints, monads, Kleisli and Eilenberg-Moore categories, co/limits, Lawvere theories, monoidal categories, operads, ...) and linear logic, even though the most important notions that are assumed will often be recalled (cf. appendix C for example). Good introductions to category theory are [Lan78], [Rie17], and [Lei16] for example.
Conventions and abbreviations: "iff" will be the abbreviation of "if and only if", "resp." of "respectively". Non-standard notations will be introduced when used for the first time, but for convenience, a glossary of notations/abbreviations can be found in appendix A.

1. Introduction

1.1 The discrete distribution monad

This report builds on Richard Garner's key observation in [Gar18] that the finite/discrete[1] distribution monad $\mathcal{D}_M\colon \mathbf{Set} \to \mathbf{Set}$ is *linear exponential*, where $M := [0,1] \subseteq \mathbb{R}$ and $\mathcal{D}_M$ is defined – if you read $\varovee$ in the following as the usual sum in $[0,1]$ – as:

Definition 1.1 — Discrete distribution monad: $\mathcal{D}_M\colon \mathbf{Set} \to \mathbf{Set}$ over M is given

- on an object $X \in \mathbf{Set}$ by:

$$\mathcal{D}_M(X) := \left\{\phi\colon X \to M \;\middle|\; \mathrm{supp}(\phi) \text{ finite and } \bigovee_{x\in X} \phi(x) = 1\right\} = \left\{\sum_{i=1}^{n} r_i \,|x_i\rangle \;\middle|\; x_i \in X,\, r_i \in M,\, \bigovee_{i} r_i = 1\right\}$$

  where $\mathrm{supp}(\phi) \subseteq X$ is the support of ϕ (set of elements $x \in X$ such that $\phi(x) \neq 0$). Such maps ϕ can be regarded as formal convex sums $\sum_{x\in X} \phi(x)\,|x\rangle$, where Dirac's ket notation is nothing but syntactic sugar drawing a distinction between elements $x \in \mathrm{dom}\,\phi$ and their occurrences in the formal sum. By convention, in these formal convex sums, $r_1\,|x\rangle + r_2\,|x\rangle$ will be identified with $(r_1 \varovee r_2)x$.
- on a morphism $f\colon X \to Y$ by:

$$\mathcal{D}_M(f) := \begin{cases} \mathcal{D}_M(X) & \longrightarrow \mathcal{D}_M(Y) \\ \displaystyle\sum_{i=1}^{n} r_i\,|x_i\rangle & \longmapsto \displaystyle\sum_{i=1}^{n} r_i\,|f(x_i)\rangle \end{cases}$$

The unit $\eta_X\colon X \to \mathcal{D}_M(X)$ and multiplication $\mu_X\colon {\mathcal{D}_M}^2(X) \to \mathcal{D}_M(X)$ of the monad are defined as:

$$\eta_X(x) := 1\,|x\rangle \qquad\qquad \mu_X\Big(\sum_{i=1}^{n} r_i\,|\phi_i\rangle\Big) := \sum_{x\in X}\Big(\bigovee_{i=1}^{n} r_i \cdot \phi_i(x)\Big)\,|x\rangle$$

What is meant by *linear exponential* is that is that $\mathcal{D}_M$ "lifts"[2] a certain tensor $\star_{(0,1)}$ making $(\mathbf{Set}, \star_{(0,1)})$ a symmetric monoidal category to a coproduct in the category of $\mathcal{D}_M$-algebras. Now, why is it interesting? For starters, Garner shows that linear exponential monads defined on symmetric monoidal categories (SMC) are the perfect setting for what Jacobs calls 'hypernormalisation' in [Jac17b], *viz.* a categorically well-behaved and totally defined generalisation of normalisation of subprobability distributions. But it does not stop here: quite the contrary, it raises a handful of compelling questions.

1.1.1 What makes $\mathcal{D}_M$ a monad

First, the only properties of $[0,1]$ that make $\mathcal{D}_M$ a monad can be summed up in the fact that it forms an **effect algebra** with a multiplication compatible with the addition, called an **effect monoid**. Effect algebras have been used since the 1990s [FB94], notably in quantum mechanical foundations, and can be thought of as merging both probabilities and propositions in one mathematical structure. More precisely, orthomodular lattices are to quantum logic [Sta15] – the logic of boolean observables in quantum mechanics, inaugurated by Birkhoff and Von Neumann in [BV36] – what boolean algebras (resp. Heyting algebras) are to classical logic (resp. intuitionistic logic). And both orthomodular lattices and the set of so-called **quantum effects**, *i.e.* self-adjoint operators (corresponding to observables) on a Hilbert space between 0 and id, are special cases of effect algebras. Now, a natural question one may ask would be: if we generalise $[0,1]$ to a general effect monoid M, resulting in what we will refer to as the **generalised distribution monad** $\mathcal{D}_M$: *under what conditions does $\mathcal{D}_M$ remain linear exponential*?

[1] we will omit the adjectives 'finite'/'discrete' in the sequel
[2] in a sense which is made precise in Garner's article

1.1.2 What makes $\mathcal{D}_M$ linear exponential

Secondly, as suggested by the name, linear exponential monads (resp. comonads) are of paramount importance in linear logic: modelling the exponential modality ? (resp. !), they are one of the three ingredients to have a model of Classical Linear Logic (CLL) along with a $*$-autonomous category (multiplicative fragment) which has finite products and coproducts (additive fragment), see for example [De 14; Mel03; Mel; Sch]. As a result, one may then wonder *if $\mathcal{D}_M$ gives rise to a model of linear logic*, for $M = [0,1]$ or even a general effect monoid.

But closer investigation of Garner's work in [Gar18] reveals more curious aspects. The reasons that enabled him to show that $\mathcal{D}_{[0,1]}$ is linear exponential are chiefly twofold.

1. On the one hand, the tensor $\star_{(0,1)}$ defined as $A \star_{(0,1)} B \ := \ A + (0,1) \times A \times B + B$ makes $(\mathbf{Set}, \star_{(0,1)})$ symmetric monoidal owing to $(0,1) = [0,1] \backslash \{0,1\}$ being a symmetric **tricocycloid** in the symmetric monoidal category (SMC) $(\mathbf{Set}, \times)$, a notion coined by Ross Street in [Str98]. The fact that $(0,1)$ is a tricocycloid has a neat geometric interpretation as a transformation of coordinates in the Euclidean plane, that we will see later. But more generally, in his paper ([Str98, Proposition 2.1.]), Street shows that, given a tricocycloid $H \in \mathscr{C}$ in a braided monoidal category $(\mathscr{C}, \otimes)$, the tensor product given by $A \star'_H B \ := \ H \otimes A \otimes B$ endows $\mathscr{C}$ with a semi-monoidal structure (*i.e.* monoidal without unit). And as pointed out by Garner, provided $\mathscr{C}$ has finite coproducts $+$ over which $\otimes$ distributes, we can co-universally make $(\mathscr{C}, \star_H)$ a monoidal category (with the initial object 0 as unit) by setting $A \star_H B \ := \ A + H \otimes A \otimes B + B$. However, H being a tricocycloid is not only a sufficient condition, it becomes necessary as soon as we require the associator of the monoidal structure induced by $\star_H$ be strong in each component, which is a natural condition to ask in the $(0,1)$ case.
   Tricocycloids are quantum algebraic objects generalising Hopf algebras (bialgebras with an antipode, the typical example thereof being group algebras over a ring), of which quantum groups are particular examples (for a brief introduction, see [Maj06]). More precisely:

   Definition 1.2 A **tricocycloid** in a symmetric monoidal category $(\mathscr{C}, \otimes)$ is an object $H \in \mathscr{C}$ with an isomorphism $v \colon H \otimes H \to H \otimes H$ satisfying the so-called *3-cocycle* condition: $(v \otimes 1)(1 \otimes \sigma)(v \otimes 1) = (1 \otimes v)(v \otimes 1)(1 \otimes v)$, where σ is the symmetry of the category and 1 the identity morphism (the subscripts are omitted). A **symmetry** for a tricocycloid H is an involution $\gamma \colon H \to H$ such that $(1 \otimes \gamma)v(1 \otimes \gamma) = v(\gamma \otimes 1)v$. A tricocycloid with a symmetry is called a *symmetric tricocycloid*.

   Beside the fact that the *3-cocycle* condition satisfied by a tricocycloid is strikingly reminiscent of the *Yang-Baxter equation* $(v \otimes 1)(1 \otimes v)(v \otimes 1) = (1 \otimes v)(v \otimes 1)(1 \otimes v)$ appearing, for instance, in quantum mechanical many-body problems (see [Jim89]), one may wonder *how such an unforeseen cocycle condition, stemming from non-abelian cohomology (see [Str87]), is related to the problem we are considering.*

2. On the other hand, the tensor $\star_{(0,1)}$ was shown to be lifted to a coproduct in the category $\mathcal{EM}(\mathcal{D}_{[0,1]})$ of $\mathcal{D}_{[0,1]}$-algebras due to $\mathcal{EM}(\mathcal{D}_{[0,1]})$ being isomorphic to the category $\mathrm{Conv}_{[0,1]}$ of (abstract) convex sets over $[0,1]$, which enabled Garner to give an explicit description of the coproduct in $\mathcal{EM}(\mathcal{D}_{[0,1]})$. But, apart from the expectation monad – also proved to be linear exponential in Garner's article – *what about other probability monads*, such as the (continuous) Giry/Radon/Kantorovitch monads (described in [Jac18] for example)? For these, even though their category of algebras can be shown to be bicomplete (complete and cocomplete) due to their being commutative (see [Jac18; Koc71]), this trick seems hardly helpful, as it heavily relies on the isomorphism $\mathrm{Conv}_{[0,1]} \cong \mathcal{EM}(\mathcal{D}_{[0,1]})$, which looks specific to the monad $\mathcal{D}_{[0,1]}$ at first glance. But it turns out that the Kleisli categories of all these other probability monads have been shown to be **effectuses** by Bart Jacobs in [Jac18]. Effectus theory [Cho+15; Jac15] is a young branch of categorical logic, developed over the past few years by Bart Jacobs, Kenta Cho and Bas and Abraham Westerbaan, whose objects of study are categorical models – called effectuses – aiming to capture the fundamentals of discrete, continuous and quantum logic and probability. To some extent, it can be informally argued that effectus theory is to quantum logic what topos theory is to intuitionistic logic. In broad terms, an effectus $\mathbb{B}$ is a category with a terminal object 1 and finite coproducts $+$ which satisfies mild pullback assumptions and a joint monicity requirement ensuring that, for all $X \in \mathbb{B}$

- the set $M_\mathbb{B} := \mathrm{Hom}_\mathbb{B}(1, 2)$ of **scalars** forms an *effect monoid* (where $2 := 1 + 1$)
- the set $\mathrm{Stat}(X) := \mathrm{Hom}_\mathbb{B}(1, X)$ of **states** of X forms an abstract convex set over $M_\mathbb{B}$
- the set $\mathrm{Pred}(X) := \mathrm{Hom}_\mathbb{B}(X, 2)$ of **predicates** over X forms an *effect module* over $M_\mathbb{B}$, *i.e.* an effect algebra which is a module over the effect monoid of scalars $M_\mathbb{B}$

Typical examples of effectuses are, among others: Set (modelling classical computation and logic), the opposite category $\mathbf{C}^*_{\mathrm{PU}}{}^{\mathrm{op}}$ of C^*-algebras and positive unital maps (modelling quantum computation/logic), but also the Kleisli categories of the aforementioned probability monads, the opposite categories $\mathcal{DL}^{\mathrm{op}}$ of distributive lattices, $\mathcal{BA}^{\mathrm{op}}$ of boolean algebras, and $\mathbf{Rng}^{\mathrm{op}}$ of rings. In 2019, Octavio Zapata proved in [Zap] that $\mathcal{K}\ell(\mathcal{D}_M)$ is an effectus as well if M is an arbitrary effect monoid. In an effectus $\mathbb{B}$, if $\omega\colon 1 \to X$ is a state and $p\colon X \to 2$ is a predicate, we can define the scalar $\omega \vDash p := p \circ \omega\colon 1 \to 2$ called **logical validity**. Depending on $\mathbb{B}$, such logical validity scalars take various forms (see appendix E.4 and [Jac15] for a more detailed exposition), ranging from membership to a subset when $\mathbb{B} = \mathbf{Set}$, to an expected value when $\mathbb{B} = \mathcal{K}\ell(\mathcal{D}_{[0,1]})$ and the trace of ωp when $\mathbb{B} = \mathbf{C}^*_{\mathrm{PU}}{}^{\mathrm{op}}$, ω is seen as a density matrix, p a quantum effect, and $X \in \mathbb{B}$ is the C^*-algebra of bounded linear operators on a finite-dimensional Hilbert space $\mathscr{H}$. The last example is an instance of the *Born rule* [Lan09] in quantum mechanics. Now, the particularly relevant property of effectuses for the problem we have at hand is that an effectus $\mathbb{B}$ induces what Jacobs calls a *state-and-effect triangle* (he gives a systematic way to construct such triangles in [Jac17a]):

$$\begin{tikzcd}
(\mathcal{EMod}_{M_\mathbb{B}})^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{Hom}_{\mathcal{EMod}_{M_\mathbb{B}}}(-,M_\mathbb{B})"] \arrow[rr, phantom, "\top"] & & \mathrm{Conv}_{M_\mathbb{B}} \cong \mathcal{EM}(\mathcal{D}_{M_\mathbb{B}}) \arrow[ll, bend left, "\mathrm{Hom}_{\mathrm{Conv}_{M_\mathbb{B}}}(-,M_\mathbb{B})"] \\
& \mathbb{B} \arrow[ul, bend left, "\mathrm{Pred} := \mathrm{Hom}_\mathbb{B}(-,1+1)"] \arrow[ur, bend right, "\mathrm{Stat} := \mathrm{Hom}_\mathbb{B}(1,-)"'] &
\end{tikzcd} \tag{1.1}$$

where $\mathcal{EMod}_{M_\mathbb{B}}$ is the category of effect modules over $M_\mathbb{B}$ and $\mathrm{Conv}_{M_\mathbb{B}}$, the category of convex sets over $M_\mathbb{B}$, comes into play!

So far, we have teased out two approaches to address the question of whether the considered monads (*e.g.* the generalised distribution monad $\mathcal{D}_M$, or another probability monad) are linear exponential, that we may refer to as *bottom-up*: starting from a tricocycloid in the base category and trying to show that the induced tensor is lifted to a coproduct in the category of algebras; and *top-down*: giving an explicit description of the coproduct in the category of algebras, and trying to see if it comes from a lifted tensor from the base category. The effectus perspective is promising with respect to the top-down approach, insofar as it involves the category of convex sets, in which coproducts are more neatly expressed. But a very natural question arising from this is: how about starting with an effectus $\mathbb{B}$, considering the generalised distribution monad $\mathcal{D}_{M_\mathbb{B}}$ over its scalars, and then forming the effectus $\mathcal{K}\ell(\mathcal{D}_{M_\mathbb{B}})$: *how do $\mathbb{B}$ and $\mathcal{K}\ell(\mathcal{D}_{M_\mathbb{B}})$ compare?*

1.1.3 Quantum-related notions

Oddly enough, we have come across several notions related to quantum logic/computation/mechanics originating from seemingly unrelated considerations: linear logic, due to being the internal logic of symmetric monoidal closed categories, can be argued to be the proper embodiment of quantum logic (see [Gir03; Pra92]), effect algebras are central in quantum mechanical foundations, tricocycloids are quantum algebraic objects, and effectuses are models of quantum logic/computation. More precisely, consider the interval $[0, 1]$:

1. for $\mathcal{D}_{[0,1]}$ to be a monad, $[0, 1]$ is an effect monoid (related to quantum mechanics/logic)
2. for $\mathcal{D}_{[0,1]}$ to be linear exponential, $(0, 1)$ is a tricocycloid (related to quantum algebra)
3. $\mathcal{K}\ell(\mathcal{D}_{[0,1]})$ is an effectus (related to quantum logic/computation) whose effect monoid of scalars is $[0, 1]$ (and $\mathcal{EM}(\mathcal{D}_{[0,1]}) \cong \mathrm{Conv}_{[0,1]}$ in the state-and-effect triangle of $\mathcal{K}\ell(\mathcal{D}_{[0,1]})$)

How do these 3 notions relate?

2. Effect Monoids vs. Tricocycloids

The goal of this section is to draw a link between effect monoids – which carry the structure required on $[0,1]$ to make $\mathcal{D}_{[0,1]}$ a monad – and tricocycloids in **Set** – which underlie the reason why $(\mathbf{Set}, \star_{(0,1)})$ can be endowed with a symmetric monoidal structure rendering $\mathcal{D}_{[0,1]} \colon (\mathbf{Set}, \star_{(0,1)}) \to \mathcal{EM}(\mathcal{D}_M)$ linear exponential. It turns out that one can come up with suitable conditions to impose on our effect monoids and tricocycloids in **Set** to go from one to the other, namely that the effect monoids have *normalisation*, and the tricocycloids be *left and double cancellative*. As it happens, imposing these extra-requirements results in an isomorphism of categories. The proofs will rarely be provided for lack of space, but they can be found in appendix D.

2.1 Effect monoids with normalisation

2.1.1 Effect algebras

As previously mentioned, effect algebras play a key role in quantum mechanical foundations and quantum logic, see [FB94; Kup; Sta15; Wil17], generalising both the structure of boolean quantum observables and quantum effects. The archetype of an effect algebra is the interval $[0,1]$, which can be thought of as a commutative monoid where $+$ is partially defined – $a, b \in [0,1]$ can be summed iff $a + b \in [0,1]$, and when it happens, they are called *orthogonal* – and where there is an element 1 enabling us to define an *orthocomplement* $a^\perp := 1 - a$ for every $a \in [0,1]$, that can be understood as the "maximally orthogonal" (*i.e.* there is no other greater orthogonal element) to a.

Definition 2.1 — A partial commutative monoid (PCM) is a set M equipped with a *zero* element $0 \in M$ and a partial binary *sum* operation $\ovee \colon M \times M \rightharpoonup M$ which is associative ($\forall x, y, z.\, (x \ovee y) \ovee z = x \ovee (y \ovee z)$), commutative ($\forall x, y.\, x \ovee y = y \ovee x$), and satisfies the unit law ($\forall x.\, 0 \ovee x = x$).

NB With the following abuse of notation in equations involving $\ovee$: the left-hand side is defined iff the right-hand side is, and when it happens, they are equal (Kleene equality).

In a PCM, $x_1, \ldots, x_n \in M$ are said to be **orthogonal** if $\bigovee_{i=1}^{n} x_i$ is defined. Two elements $x, y \in M$ being orthogonal is denoted by $x \perp y$. A morphism of PCMs is a function between the underlying sets preserving $\ovee$. PCM and their morphisms form a category $\mathcal{PCM}$.

Definition 2.2 — An effect algebra is a PCM E with an *orthocomplement* function $(-)^\perp \colon E \to E$ such that: 0 is the unique element orthogonal to $1 := 0^\perp$ and for all $x \in E$, $x^\perp$ is the unique element (orthocomplement) satisfying $x \ovee x^\perp = 1$.

NB An effect algebra carries a poset structure by setting $x \leq y \iff \exists z; \quad x \ovee z = y$. A *partial difference* can also be given by $y \ominus x = z \iff x \ovee z = y$.

Effect algebras and functions between the underlying sets preserving $\ovee$ and 1 yield a subcategory $\mathcal{EA}$ of $\mathcal{PCM}$. Effect algebras can be thought of as generalising probabilities and propositions.

Example 2.1 — Non-examples and examples of effect algebras
By abuse of notation, in the following examples, one denotes in the same way the sum operations and their restriction to the subdomain of couples of orthogonal elements. To further illustrate the idea, we give examples of PCMs that are *not* effect algebras in appendix B.

- The one-element and two-elements sets are, respectively, the terminal and initial objects of $\mathcal{EA}$
- Interval effect algebras:
  - the ultimate example of an effect algebra is the interval $[0,r] \subseteq \mathbb{R}$, for $r \in \mathbb{R}$ ($r = 1$ will play a key role later), with $x \perp y \iff x + y \leq r$, $\ovee := +$, $(-)^\perp := r - (-)$

- similarly, the bounded linear operators $U : \mathscr{H} \to \mathscr{H}$ on a Hilbert space $\mathscr{H}$ that are positive and below the identity (called *effects*, whence the name "effect algebra") form an effect algebra too

- Orthomodular lattices: an **ortholattice** L is a bounded (with least element 0 and greatest element 1) lattice (poset where every two $x, y \in L$ have a greatest lower bound $x \wedge y$ and least upper bound $x \vee y$) equipped a complementation $(-)^{\perp}$ ($x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$) which is involutive ($x^{\perp\perp} = x$) and order-reversing ($x \leq y \implies y^{\perp} \leq x^{\perp}$). An **orthomodular lattice** is an otholattice where the modular law holds: $x \leq y \implies y = x \vee (x^{\perp} \wedge y)$. With $x \perp y \iff x \leq y^{\perp}$, $\ovee \; := \; \vee$, orthomodular lattices are effect algebras. Special cases of orthomodular lattices of particular importance are:
  - Boolean algebras (distributive orthomodular lattices), *e.g.* the boolean algebra of measurable subsets of a measure space (where $U \perp V \iff U \cap V = \emptyset$ and $\ovee = \cup$). Note, as it happens, that a probability measure $\mu \colon \Sigma \to [0,1]$ is nothing but a morphism of effect algebras. The category $\boldsymbol{\mathcal{BA}}$ of boolean algebras is a full subcategory of $\boldsymbol{\mathcal{EA}}$ (see [Jac15, Lemma 2.3.]).
  - The lattice of closed subspaces of a Hilbert space, which is of paramount significance in quantum logic. In their seminal paper [BV36], Birkhoff and von Neumann gave birth to quantum logic by pointing out that the closed subspaces of a Hilbert space, which form an orhomodular lattice, can be thought of as representing quantum "experimental propositions"[a] about physical observables. What is meant by that is that such quantum propositions are *semantically interpreted*[b] by closed subspaces of the phase-space of the considered quantum system, which is a Hilbert space, in von Neumann's mathematical foundations of quantum mechanics [NW18].

[a] subsets of the observation-space, the space of the readings from measurements of a given physical (*viz.* quantum) system
[b] to use an anachronistic computer science/logic terminology, which wasn't used by von Neumann

2.1.2 Effect monoids and modules

In [JM12], Jacobs and Mandemaker proved that $\boldsymbol{\mathcal{EA}}$ is bicomplete and symmetric monoidal, where morphisms $f \in \mathrm{Hom}_{\boldsymbol{\mathcal{EA}}}(E_1, E_2)$ correspond to bihomomorphisms $\tilde{f} \colon E_1 \times E_2 \to D$, *i.e.* functions between the underlying sets which are homomorphisms of PCMs separately in each coordinate, and such that $\tilde{f}(1,1) = 1$. The tensor unit is the initial object (the two-elements effect algebra). An **effect monoid** is a monoid in the category $\boldsymbol{\mathcal{EA}}$, *i.e.* an effect algebra $M \in \boldsymbol{\mathcal{EA}}$ equipped with an associative *multiplication* $\cdot \colon M \times M \to M$ distributing over the partial sum $\ovee$ and having $1 \in M$ as neutral element. In a standard way, effect monoids form a subcategory $\boldsymbol{\mathcal{EMon}}$ of $\boldsymbol{\mathcal{EA}}$. For an effect monoid $M \in \boldsymbol{\mathcal{EMon}}$, it is well-known (see [nLab]) that the endofunctor $M \otimes (-) \colon \boldsymbol{\mathcal{EA}} \to \boldsymbol{\mathcal{EA}}$ is a monad. The category $\boldsymbol{\mathcal{EMod}}_M$ of effect modules over M is the category $\mathcal{EM}(M \otimes (-))$ of algebras of this monad; that is, the category of actions of the monoid M. An **effect module** over M can be more concretely described as an effect algebra E endowed with a scalar multiplication $\cdot \colon M \otimes E \to E$ preserving 0 and $\ovee$ in each coordinate and satisfying $1 \cdot e = e$ and $r \cdot (s \cdot e) = (r \cdot s) \cdot e$.

Let M be an effect monoid.

Proposition — D.1 For all $a, b \in M$, $a^{\perp}b^{\perp} = (a \ovee a^{\perp}b)^{\perp} = (b \ovee ab^{\perp})^{\perp}$

M will be said to have **normalisation** iff $\forall a \neq 1, b \in M.\, a \perp b \implies \exists! c;\ b = a^{\perp}c$. This notion brings us closer to tricocycloids, since, as previously mentioned, $(0,1)$ being a tricocycloid leads the induced tensor to be lifted to a coproduct in the category of $\mathcal{D}_{[0,1]}$-algebras, which is to be construed as a suitable setting for a generalisation of normalisation of subprobabilities [Gar18].

Corollary — D.1 If M has normalisation, M is *left-cancellative away from zero*, *i.e.*

$$\forall a \neq 0, b, b' \in M. \quad ab = ab' \quad \implies \quad b = b'$$

Proposition — D.2 If M has normalisation, then for all $a \ovee b_1 \ovee \cdots \ovee b_n = 1$ with $a \neq 1$, there exist unique $c_1 \ovee \cdots \ovee c_n = 1$ such that $\forall i.\, b_i = a^{\perp}c_i$

2.2 Tricocycloids in Set

Recall the general definition 1.2 of a tricocycloid in a braided monoidal category. A **lax tricocycloid** is defined like a tricocycloid, except that v is no longer required to be invertible. We have the following characterisation:

Lemma — D.2. Let (H, v, γ) be a lax tricocycloid with a symmetry. It is a tricocycloid iff $v(\gamma\otimes\gamma)v(\gamma\otimes\gamma) = 1$, in which case $v^{-1} = (\gamma\otimes\gamma)v(\gamma\otimes\gamma)$.

From the definition 1.2, we can read off, when this category is $(\mathbf{Set}, \times)$ (which we will assume from now on unless specified otherwise):

Definition 2.3

A (lax) **tricocycloid** in $(\mathbf{Set}, \times)$ is given by a set H and a (*bijective*) function

$$v\colon \begin{cases} H\times H \longrightarrow H\times H \\ (r,s) \longmapsto (r\cdot s, r\diamond s) \end{cases} \quad \text{satisfying:}$$

(i) *Associativity*: $\cdot$ is associative

(ii) *3-cocycle 1*: $(r\diamond st)(s\diamond t) = rs\diamond t$

(iii) *3-cocycle 2*: $(r\diamond st)\diamond(s\diamond t) = r\diamond s$

A **symmetry** for a tricocycloid $H\in\mathbf{Set}$ is an *involution* $\gamma\colon \begin{cases} H\longrightarrow H \\ r\longmapsto r^\perp \end{cases}$ such that:

(iv) *Symmetry 1*: $r\cdot s^\perp = (r\cdot s)^\perp(r\diamond s)$

(v) *Symmetry 2*: $(r\diamond s^\perp)^\perp = (r\cdot s)^\perp\diamond(r\diamond s)$

NB We write $\cdot$ as juxtaposition, and allow it to bind more tightly than $\diamond$.

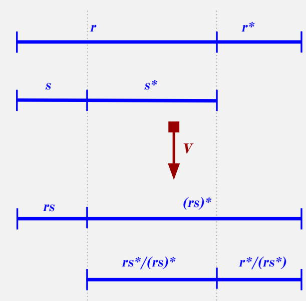
Example 2.2 — The open interval $(0,1)$.

The prime example of a symmetric tricocycloid in Set is the open interval $(0,1)$, as shown by Garner in [Gar18], for which $r\cdot s := rs$, $r\diamond s := \frac{rs^*}{(rs)^*}$ where $(-)^* := 1-(-)$, and $\gamma := (-)^*$. The fact that it is a symmetric tricocycloid has a neat geometric interpretation as a transformation of coordinates in the Euclidean plane: $v(r,s) = (r\cdot s, r\diamond s)$ turns, in $\mathrm{Conv}_{[0,1]} \cong \mathcal{EM}(\mathcal{D}_{[0,1]})$, the formal convex combination

$$r\big(s\,|A\rangle + s^*\,|B\rangle\big) + r^*\,|C\rangle$$

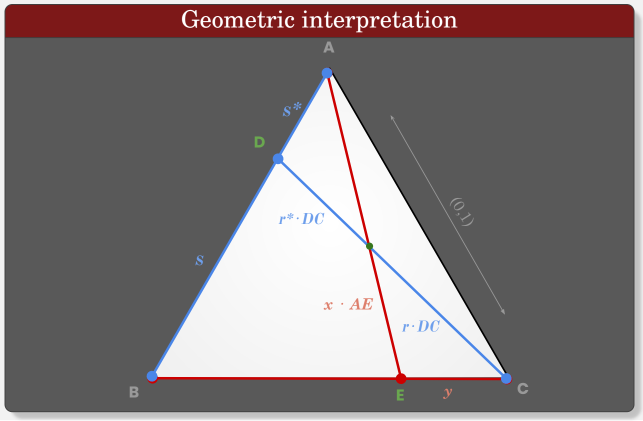
into the convex combination

$$rs\,|A\rangle + (rs)^*\Big(\frac{rs^*}{(rs)^*}\,|B\rangle + \frac{r^*}{(rs)^*}\,|C\rangle\Big) = r\cdot s\,|A\rangle + (r\cdot s)^\perp\big((r\diamond s)\,|B\rangle + (r\diamond s)^\perp\,|C\rangle\big)$$



This can be shown geometrically (cf. appendix H for a detailed proof): in the following drawing, where ABC is equilateral (each side may be thought of as a copy of $H = (0,1)$), v sends (r,s) to (x,y):

Geometric interpretation



The following lemma is a useful adjunct to D.2, giving another equivalent condition for a lax tricocycloid to be a tricocycloid, provided it is in Set.

Lemma 2.1 Let (H, v, γ) be a lax tricocycloid with symmetry in Set. It is a tricocycloid iff we have

(vi) *Symmetry 1 orthogonal*: $(rs)^\perp (r \diamond s)^\perp = r^\perp$ (vii) *Symmetry 2 orthogonal*: $(rs)^\perp \diamond (r \diamond s)^\perp = s^\perp$

Proof

We have that: $(r, s) \xmapsto{v} (rs, r \diamond s) \xmapsto{\gamma \otimes \gamma} \big((r \cdot s)^\perp, (r \diamond s)^\perp\big) \xmapsto{v} \big((rs)^\perp (r \diamond s)^\perp, (rs)^\perp \diamond (r \diamond s)^\perp\big)$. So these equations hold iff $\forall r, s.\ (v(\gamma \otimes \gamma)v)((r, s)) = (\gamma \otimes \gamma)((r, s))$, that is: v is invertible by Lemma D.2. ■

A tricocycloid will be said to be **left-cancellative** if $\forall r, s, s'.\ rs = rs' \implies s = s'$ and satisfying the **double cancellation property**/be double cancellative if $\forall r, s, r', s'.\ rs = s'r'$ and $rs^\perp = (s')^\perp r' \implies r = r'$. A tricocycloid where both of these laws hold will be called **left/double cancellative**.

We will now go about proving the fact that the categories $\mathcal{EMonNorm}$ of effect monoids with normalisation and $\mathcal{TricoCanc}$ of symmetric double/left cancellative tricocycloids are isomorphic, thereby answering some of our introductory questions.

2.3 From a tricocycloid to an effect monoid

This direction is the trickiest one, but it is the one that elucidates the constraints we need to impose on our tricocycloids to have a correspondence with effect monoids.

Lemma — D.3. In a symmetric tricocycloid $H \in$ **Set**, for all $r, s, t, d, d' \in H$, we have:

(vi) $(rs)^\perp (r \diamond s)^\perp = r^\perp$ (viii) $(r \diamond st)(s \diamond t)^\perp = (rs \diamond t)^\perp (r \diamond s)$ (x) $((s^\perp \diamond d')^\perp d)^\perp = ((sd)^\perp \diamond (s \diamond d)^\perp d')^\perp$

(vii) $(rs)^\perp \diamond (r \diamond s)^\perp = s^\perp$ (ix) $(r \diamond st) \diamond (s \diamond t)^\perp = ((rs \diamond t)^\perp \diamond (r \diamond s))^\perp$ (xi) $((s^\perp \diamond d')^\perp \diamond d)^\perp = (s \diamond d)^\perp \diamond d'$

The previous lemma enables us to prove the following corollary and proposition, which will be key in showing right distributivity of the effect monoid constructed out of a given symmetric left/double cancellative tricocycloid.

Corollary — D.4 In a symmetric tricocycloid, for all s, d:

$$(s^\perp d)^\perp ((s^\perp \diamond d)^\perp d)^\perp = (sd)^\perp ((s \diamond d)^\perp d)^\perp$$

Proposition — D.3 In a symmetric tricocycloid where the double cancellation property holds:

$$\forall d, s.\ \ d^\perp = (sd)^\perp ((s \diamond d)^\perp d)^\perp$$

Let $H \in$ **Set** be a symmetric left-cancellative tricocycloid satisfying the double cancellation property, and define $M := \overline{H} := H \cup \{0, 1\}$, where 0 and 1 are two extra elements, and $\cdot, \diamond \colon H \times H \to H, (-)^\perp \colon H \to H$ are extended to partial functions on $\overline{H} := H \cup \{0, 1\}$ as follows, for all $x \in \overline{H}$:

$1 \cdot x = x \cdot 1 = x$	$1 \diamond x = 1$ if $x \neq 1$, else not defined		$0 \diamond x = 0$
$0 \cdot x = x \cdot 0 = 0$	$x \diamond 1 = 0$	$0^\perp = 1$	$x \diamond 0 = x$

We also put $a \perp b \overset{\text{def}}{\iff} \exists r, s. \begin{cases} a = rs \\ b = rs^\perp \end{cases}$ and $\overbrace{a}^{=rs} \perp \overbrace{b}^{=rs^\perp} \implies a + b := r$. Note that this (r, s) couple is unique by injectivity of v, since:

$$\begin{cases} rs = r's' \\ \underbrace{rs^\perp}_{(rs)^\perp(r\diamond s)} = \underbrace{r'(s')^\perp}_{(r's')^\perp(r'\diamond s')} \end{cases} \overset{\substack{\text{left cancellativity} \\ \downarrow}}{\Longrightarrow} \begin{cases} rs = r's' \\ (r\diamond s) = (r'\diamond s') \end{cases}$$

NB If we put $1 \diamond 1 = 0$ (it's either this or 1, to comply with at least one of the conflicting equations we have), $H \cup \{0,1\}$ is a lax tricocycloid, and Street's construction ([Str98, Proposition 2.1.]) leads to a lax semi-monoidal category. The same holds, more generally, for every effect monoid M.

Theorem 2.2 — From a tricocycloid to an effect monoid. Every *symmetric left-cancellative* tricocycloid H *satisfying the double cancellation property* gives rises to an effect monoid $\overline{H} \ := \ H \cup \{0,1\}$ having normalisation, with the operations/relations defined above. ■

Proof

We give a sketch of the proof. It has been implemented in details in Why3 (cf. appendix I.1 and the dedicated GitHub repository).

- *PCM structure:* the commutativity of $+$ stems from $(-)^\perp$ being involutive (if (r,s) makes (a,b) orthogonal, $(r,s^\perp)$ do the same for (b,a)), and the unit law for $a \perp 0$ clearly holds with $r = a,\ s = 1$. The associativity deserves to be expounded on a bit more: if $rs = a \perp b = rs^\perp$ and $r's' = a + b \perp c = r'(s')^\perp$, then

$$a = r'(s's) \qquad b = r's's^\perp \overset{(iv)}{=} r'(s's)^\perp(s'\diamond s) \qquad c = r'(s')^\perp \overset{(vi)}{=} r'(s's)^\perp(s'\diamond s)^\perp$$

  Therefore $b \perp c,\ b + c \ := \ r'(s's)^\perp \perp r'(s's) = a$, and $a + (b+c) \ := \ r' = (a+b)+c$.
- *Effect algebra structure:* Existence of the orthocomplement is easily shown, and 0 is the unique orthocomplement of 1 (since $1 = rs \Longrightarrow r = 1$ and $s = 1$, so $rs^\perp = r \cdot 0 = 0$). As for uniqueness of the orthocomplement: if $x = rs = r's'$ and $y = rs^\perp$ and $y' = r'(s')^\perp$ with $r = x + y = 1 = x + y' = r'$, then $s = x = s'$, whence $y = y'$.
- *Effect monoid structure:* $\cdot$ is associative, preserves 0 in both variables and satisfies the unit laws by construction. Left distributivity is easily shown, by uniqueness of (r,s) in the definition of orthogonality. Right distributivity is more nettlesome: let $rs = a \perp b = rs^\perp$ and $d \in \overline{H}$. Then $ad \perp bd$ because

$$\begin{aligned} ad = rsd &\overset{(vi)}{=} r((s^\perp d)^\perp(s^\perp \diamond d)^\perp)d = r(s^\perp d)^\perp((s^\perp\diamond d)^\perp d) \\ &\overset{(iv)}{=} r((s^\perp d)^\perp((s^\perp\diamond d)^\perp d)^\perp)^\perp((s^\perp d)^\perp \diamond ((s^\perp\diamond d)^\perp d)^\perp) \\ &\overset{\text{Corollary D.4}}{=} r\,\underbrace{((sd)^\perp((s\diamond d)^\perp d)^\perp)^\perp}_{\overset{\text{Proposition D.3}}{=} d}\underbrace{((s^\perp d)^\perp \diamond ((s^\perp\diamond d)^\perp d)^\perp)}_{\text{denoted by } x} \end{aligned}$$

  and

$$bd = rs^\perp d \overset{(vi)}{=} r\,\underbrace{((s^\perp d)^\perp((s^\perp\diamond d)^\perp d)^\perp)^\perp}_{\overset{\text{Corollary D.4}}{=} d}\underbrace{((s^\perp d)^\perp \diamond ((s^\perp\diamond d)^\perp d)^\perp)^\perp}_{x^\perp}$$

  and consequently: $ad + bd = rd = (a+b)d$
- *Normalisation:* if $1 \neq rs = a \perp b = rs^\perp$, then $(r,s) \neq (1,1)$, and $b \overset{(iv)}{=} a^\perp(r\diamond s)$, uniqueness stemming from left-cancellativity.

■

2.4 From an effect monoid to a tricocycloid

Lemma — D.5. Let H be a set equipped with a function $v\colon H \times H \to H \times H, (r,s) \mapsto (r \cdot s, r \diamond s)$ and an involution $\gamma\colon H \to H, r \mapsto r^{\perp}$. Suppose that $\cdot$ is left cancellative. Then H is a symmetric tricocycloid iff $\cdot$ is associative and the axioms *Symmetry 1* and *Symmetry 1 orthogonal* hold.

Theorem 2.3 — From an effect monoid to a tricocycloid. Every effect monoid M with normalisation can be turned into a symmetric left/double cancellative tricocycloid $\mathring{M} := M\backslash\{0,1\}$. ■

Proof
Let M be an effect monoid with normalisation, and set $H := \mathring{M} := M\backslash\{0,1\}$. For any $r, s \in H$ we have $rs \perp rs^{\perp}$, so we can define $r \diamond s$ as being the unique (by Corollary D.1) z such that $(rs)^{\perp}z = rs^{\perp}$. On top of that, in H, we define $\cdot$ to be the restricted multiplication, and the symmetry $\gamma := (-)^{\perp}$. As H is left-cancellative by Corollary D.1, it suffices to check that $\cdot$ is associative (which is immediate), and that the axioms *Symmetry 1* and *Symmetry 1 orthogonal* hold. *Symmetry 1* holds by definition of $\diamond$. As for *Symmetry 1 orthogonal*, we get from Proposition D.1 that

$$(rs)^{\perp}(r \diamond s)^{\perp} = (rs \ovee (rs)^{\perp}(r \diamond s))^{\perp} = (rs \ovee rs^{\perp})^{\perp} = r^{\perp}$$

as desired. The tricocycloid being left/double cancellative straightforwardly follows from Corollary D.1 and distributivity of $\cdot$ over $\ovee$. ■

2.5 Isomorphism of categories

Theorem 2.2 and Theorem 2.3 now yield the desired result, *viz.* the categories of effect monoids with normalisation and of symmetric tricocycloids in Set that have left and double cancellation are isomorphic.

Let $\mathscr{EMonNorm}$ be the category of effect monoids with normalisation and maps of effect algebras preserving multiplication, $\mathscr{TricoCanc}$ be the category of symmetric left/double-cancellative tricocycloids in Set and functions commuting with γ and v.

Theorem — D.6. $$\mathscr{EMonNorm} \cong \mathscr{TricoCanc}$$

We will now bring effectuses into the picture, as discussed in the introduction. First, note that we will resort to the following maps notations in the sequel (all of these are recapped in appendix A).

$:= [[\kappa_1, \kappa_2\kappa_1], \kappa_2\kappa_2]\colon (X+Y)+Z \to X+(Y+Z)$

$:= {}^{-1}\colon X+(Y+Z) \to (X+Y)+Z$

$:= (X+Y)+Z \longrightarrow X+(Y+Z) \longrightarrow (Y+Z)+X$

$:= {}^{-1}\colon (Y+Z)+X \to (X+Y)+Z$

$:= [[\kappa_1, \kappa_2], \kappa_2]\colon (X+Y)+Y \to X+Y$

$:= [[\kappa_2, \kappa_1], \kappa_2]\colon (X+Y)+X \to Y+X$

$:= [\kappa_2, \kappa_1]\colon X+Y \to Y+X$

$:= [[\kappa_1, \kappa_3], \kappa_2]\colon (X+Y)+Z \to (X+Y)+Z$

$:= [[\kappa_2, \kappa_1], \kappa_3]\colon (X+Y)+Z \to (X+Y)+Z$

Handy “shuffle maps” notations (the symbols illustrate what the maps do)

3. Effectuses

Bart Jacobs' effectuses, introduced in [Jac15], provide a categorical setting for discrete, continuous and quantum probability and logic. In appendix E, we give a handful of relevant effectus theoretic lemmas.

Definition 3.1 — An effectus $\mathbb{B}$ is a category satisfying the following properties:

- it has finite coproducts $(0, +)$ and a terminal object 1
- for all $A, B, X, Y \in \mathbb{B}, f : A \longrightarrow B, g : X \longrightarrow Y$, the following diagrams are pullbacks:

$$
\begin{tikzcd}
A + X \arrow[r, "\mathrm{id}+g"] \arrow[d, "f+\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A + Y \arrow[d, "f+\mathrm{id}"] \\
B + X \arrow[r, "\mathrm{id}+g"'] & B + Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\kappa_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\kappa_1"] \\
X + A \arrow[r, "\mathrm{id}+f"'] & X + B
\end{tikzcd}
\tag{3.1a, 3.1b}
$$

- the maps $\begin{tikzcd}(1+1)+1 \arrow[r, bend left, "\mathbb{IV} := [[\kappa_1,\kappa_2],\kappa_2]"] \arrow[r, bend right, "\mathbb{XV} := [[\kappa_2,\kappa_1],\kappa_2]"'] & 1+1\end{tikzcd}$ are *jointly monic*, *i.e.* $\forall f, g, \begin{cases} \mathbb{IV} \circ f = \mathbb{IV} \circ g \\ \mathbb{XV} \circ f = \mathbb{XV} \circ g \end{cases} \implies f = g$

We denote by $M_{\mathbb{B}} := \mathrm{Pred}(1) = \mathrm{Stat}(2) = \mathrm{Hom}_{\mathbb{B}}(1, 2)$ (where $2 = 1 + 1$) its effect monoid of **scalars** (Theorem E.6), $\mathrm{Pred} := \mathrm{Hom}_{\mathbb{B}}(-, 2) \colon \mathbb{B} \to (\mathcal{EMod}_{M_{\mathbb{B}}})^{\mathrm{op}}$ its **predicate functor** (Lemma E.11) and $\mathrm{Stat} := \mathrm{Hom}_{\mathbb{B}}(1, -) \colon \mathbb{B} \to \mathrm{Conv}_{M_{\mathbb{B}}}$ its **state functor** (Lemma E.10).

Example 3.1 We briefly give some examples of effectuses (for a more complete summarising table, see E.4, and for more details, see [Cho+15; Jac15]):

- One of the most simple examples of effectus is the category **Set** of sets and functions, modelling classical computation/logic. States $\omega \colon 1 \to X$ in **Set** are elements $\omega \in X$, predicates $p \colon X \to 2$ are subsets $p \subseteq X$. The effect monoid of scalars $M_{\mathbf{Set}} := \{0, 1\}$ is the set of boolean truth values, so that the logical validityE.2 is set membership.
- The Kleisli category of the generalised distribution monad $\mathcal{K}\ell(\mathcal{D}_M)$, where M is an effect monoid, is another important example of an effectus. States $\omega \colon 1 \to X$ in $\mathcal{K}\ell(\mathcal{D}_M)$ correspond to maps $1 \to \mathcal{D}_M(X)$, and so are generalised probability distributions $\omega \in \mathcal{D}_M(X)$ with coefficients in M, predicates $p \colon X \to 2$ correspond to maps $X \to \mathcal{D}_M(2)$, where $\mathcal{D}_M(2) \cong M$, *i.e.* 'fuzzy' predicates $p \in M^X$ valued in M. The effect monoid of scalars is M, and logical validity is the expected value.
- Another key example of effectus is the opposite category ${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$ of C^*-algebras with positive unital maps (for a quick reminder about C^*-algebras, see F), modelling quantum computation and logic (cf. [HZ08; Kup]). Note that ${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$ has terminal object (the field of complex numbers $\mathbb{C}$ is initial in $\mathbf{C}^*_{\mathrm{PU}}$) and finite coproducts ($\mathbf{C}^*_{\mathrm{PU}}$ has finite products by taking finite products of underlying sets and defining the operations pointwise). States in the effectus ${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$ coincide with the operator theoretic notion of *states*: positive unital maps $\omega \colon X \to \mathbb{C}$. Predicates on X are positive unital maps $q \colon \mathbb{C} \times \mathbb{C} \to X$ which are in one-to-one correspondence with *effects* $p \in X$ such that $0 \le p \le 1$ by setting $p := q((1, 0))$ and $q((\alpha, \beta)) = \alpha p + \beta(1 - p)$. In [MG99], the authors point out that there are two types of effects: *sharp* effects, that are projections describing accurate yes/no measurements, and *unsharp* effects, describing imprecise yes/no measurements. Scalars in $\mathbb{B} = {\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$ are effects in the C^*-algebra $\mathbb{C}$, *i.e.* the interval $[0, 1] \subseteq \mathbb{R}$. Logical validity corresponds to evaluation of ω at p, which amounts to taking the trace of ωp when X is the C^*-algebra of bounded linear operators over of a finite-dimensional Hilbert space $\mathscr{H}$, and ω is seen as a density matrix [Bec00] over $\mathscr{H}$.

For our purposes, a crucial property an effectus may have is *normalisation*, introduced by Jacobs in [JWW15].

3.1 Effectuses with normalisation

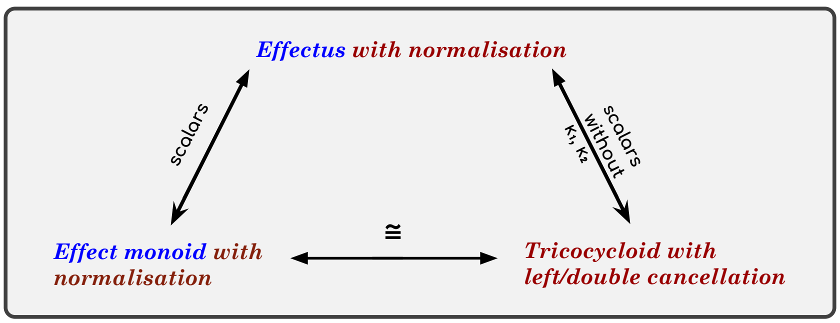
Definition 3.2 An effectus has **normalisation** iff for every map $\sigma\colon 1 \to X+1$ such that $\sigma \neq \kappa_2$, there exists a unique state $(\!|\sigma|\!)\colon 1 \to X$ such that $((\!|\sigma|\!) + \mathrm{id})\,(! + \mathrm{id})\,\sigma = \sigma$.

Example 3.2 As shown in [Cho+15; JWW15], the effectuses $\mathcal{K}\ell(\mathcal{D}_{[0,1]})$ and ${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$ have normalisation:

- In $\mathcal{K}\ell(\mathcal{D}_{[0,1]})$, a map $\sigma\colon 1 \to X+1$ is a probability subdistribution on X (*i.e.* with total measure $s := \sum_{x\in X}\omega(x) \leq 1$). If such a σ is non-zero on X (*i.e.* $\sigma \neq \kappa_2$, that is $s \neq 0$), we recover the usual notion of normalisation of probabilities: $(\!|\sigma|\!)(x) = \frac{\sigma(x)}{s}$
- In ${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$, if $\sigma\colon X \times \mathbb{C} \to \mathbb{C} \in \mathbf{C}^*_{\mathrm{PU}}$ is not equal to π_2, then $[0,1] \ni \sigma((1,0)) \neq 0$, and $(\!|\sigma|\!)(x) = \frac{\sigma((x,0))}{\sigma((1,0))}$ is indeed linear and positive unital.

These two examples are special cases of a theorem attributed to Sean Tull in [Cho+15] stating that every effectus whose effect monoid of scalars is $[0,1]$ has normalisation, and

We will now relate Jacobs' effectuses with normalisation to the notions of effect monoid with normalisation and symmetric left/right cancellative tricocycloid in **Set** we introduced before. In a nutshell, given an effectus $\mathbb{B}$ with normalisation, its effect monoid of scalars $M_{\mathbb{B}}$ has normalisation, and $M\backslash\{1,0\} = M\backslash\{\kappa_1,\kappa_2\}$ forms a symmetric tricocycloid with left/double cancellativity. We have the following situation:



3.2 From an effectus to an effect monoid

Not only does the set of scalars of an arbitrary effectus $\mathbb{B}$ form an effect monoid (E.6), this effect monoid happens to have normalisation in the sense we defined in section 2.1.2 as soon as $\mathbb{B}$ does in Jacobs' sense. As before, the proof can be found in appendix (Theorem G.1).

Theorem — Scalars of an effectus with normalisation form an effect monoid with normalisation (G.1). If $\mathbb{B}$ is an effectus with normalisation, $M_{\mathbb{B}} := \mathrm{Hom}_{\mathbb{B}}(1,2)$ is an effect monoid with normalisation.

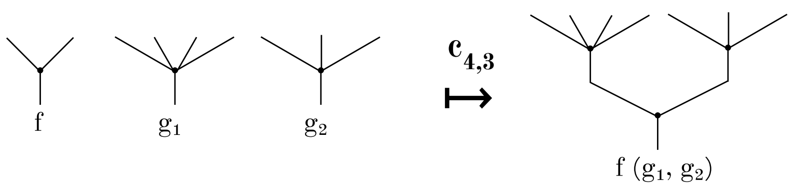
3.3 From an effectus to a tricocycloid

We will now show that for every effectus $\mathbb{B}$, the set $M_{\mathbb{B}}\backslash\{\kappa_1,\kappa_2\}$ of scalars without the scalar $1 := \kappa_1$ and the scalar $0 := \kappa_2$ forms an symmetric left/double cancellative tricocycloid in **Set** as soon as $\mathbb{B}$ has normalisation. While trying to prove this directly, I got stuck in the calculations (which quickly become very tedious), but it turns out that there is an elegant way to painlessly obtain the desired result by stating the problem in terms of *operads*, a concept that we now briefly recall[1]. Operads – a word coined by May[2] in [May89] – are

[1] for crash courses about operads, see [May97; Sta]. For more comprehensive references, see [Fre09; May89]

[2] a mix between 'operations' and 'monads'. In [May97], he says: ≪ The name "operad" is a word that I coined myself, spending a week thinking about nothing else. ≫

an abstraction of families of composable functions of various arities. They play an important role in several fields, ranging from homotopy theory (from where they originated) to homological algebra, algebraic topology, and mathematical physics. We will more specifically be concerned with *symmetric operads*, and the archetype thereof: the *endomorphism operad*. An **operad** $\mathscr{O}$ is a family of sets $\mathscr{O}_k$ (also denoted by $\mathscr{O}(k)$) for every $k \geq 0$, thought of as sets of k-ary operations. Graphically, we can depict every $f \in \mathscr{O}_k$ in a 'string-diagrammatic' way as a tree with one node (denoting the operation f), k incoming edges from above (inputs), and one edge going out below (output). Such an operation/tree f of arity k can be composed with other operations $g_1, \ldots, g_k$ of arities $n_1, \ldots, n_k$ by pasting each of the k input edges of f to the output edge of one g_i; the resulting operation $f(g_1, \ldots, g_k)$ being of arity $n_1 + \cdots + n_k$. In other words, for every $k, n_1, \ldots, n_k \in \mathbb{N}$, there is a composition map $c_{n_1,\ldots,n_k} \colon \mathscr{O}_k \times \mathscr{O}_{n_1} \times \cdots \times \mathscr{O}_{n_k} \to \mathscr{O}_{n_1+\cdots+n_k}$ sending $f, g_1, \ldots, g_k$ to $f(g_1, \ldots, g_k)$. For example, with $k = 2, n_1 = 4, n_2 = 3$:



Composition is required to be *associative*, *i.e.* $f\big(g_1(h_{1,1}, \ldots, h_{1,n_1}), \ldots, g_k(h_{k,1}, \ldots, h_{k,n_k})\big)$ equals $(f(g_1, \ldots, g_k))(h_{1,1}, \ldots, h_{1,n_1}, \ldots, h_{k,1}, \ldots, h_{k,n_k})$, and have a unary *unit* $1 \in \mathscr{O}_1$ ($1(f) = f,\ f(1, \ldots, 1) = f$). In the obvious way, we can as well define a notion of *suboperad* of an operad. An operad is said to be **symmetric** if the symmetric group $\mathfrak{S}_n$ acts on $\mathscr{O}_k$ on the right for all k ($(f\sigma)(g_1, \ldots, g_k) = \big(f(g_1, \ldots, g_k)\big)\widetilde{\sigma}$, where $\sigma \in \mathfrak{S}_k$ and $\widetilde{(-)} \colon \mathfrak{S}_k \to \mathfrak{S}_{n_1+\cdots+n_k}$ is defined in the obvious way). Graphically, this corresponds to a permutation of the input edges. Finally, note that generally, symmetric operads can be defined in the same way in any symmetric monoidal category (by replacing the cartesian product by the tensor).

Example 3.3 — The endomorphism operad End_X is a fundamental example of operad, for $X \in \mathscr{C}$ where $(\mathscr{C}, \times)$ is a cartesian category. $\mathrm{End}_X(k) := \mathrm{Hom}_{\mathscr{C}}\left(X^k, X\right)$, and composition is given by composition and product of maps: for example, if $f \colon X^2 \to X$, $g_1 \colon X^4 \to X$ and $g_2 \colon X^3 \to X$, $f(g_1, g_2) := f \circ (g_1 \times g_2) \colon X^7 \to X$. The general definition of operad is an abstraction of this very example.

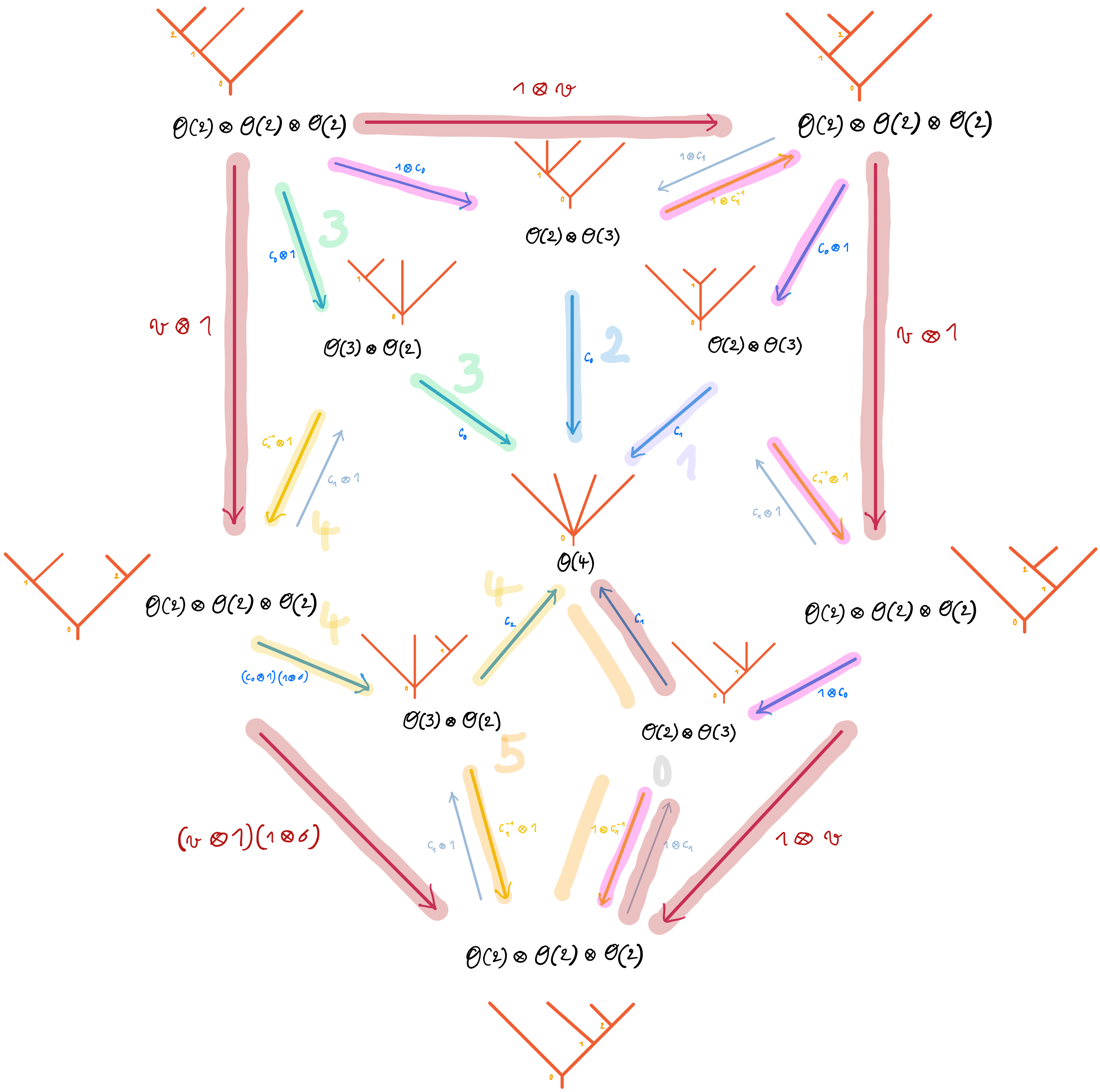
We now go back to our effectuses and tricocycloids. Given an effectus $\mathbb{B}$ with normalisation, we will apply the following observation to (a suboperad of) the endomorphism operad of $\mathbb{B}^{\mathrm{op}}$:

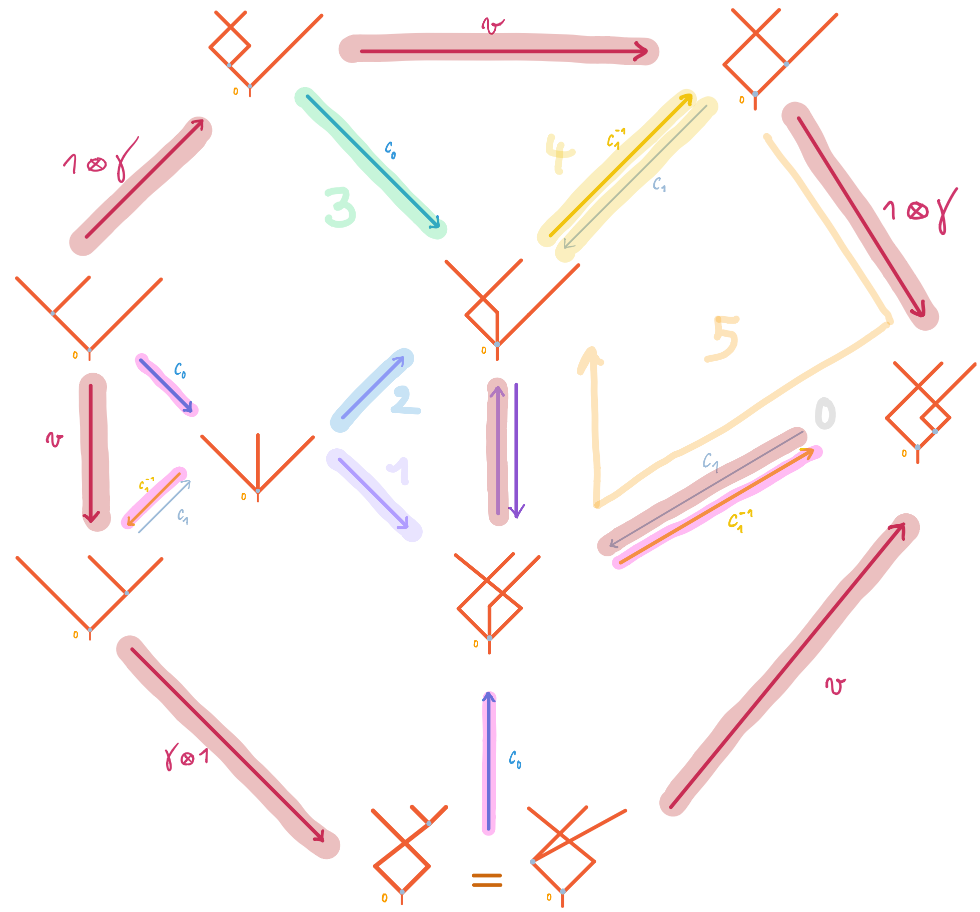
Lemma 3.1 Let $\mathscr{O}$ be a symmetric operad. To to simplify the notations, we put $c_0 := c_{2,1}(-,-,1) \colon \mathscr{O}(2) \otimes \mathscr{O}(2) \otimes \mathscr{O}(1) \to \mathscr{O}(3)$ and $c_1 := c_{1,2}(-,1,-) \colon \mathscr{O}(2) \otimes \mathscr{O}(1) \otimes \mathscr{O}(2) \to \mathscr{O}(3)$. If c_1 is invertible and $c_{1,3}(-,1,-) \colon \mathscr{O}(2) \otimes \mathscr{O}(1) \otimes \mathscr{O}(3) \to \mathscr{O}(4)$ is monic, then $(\mathscr{O}(2), v, \gamma)$ is a symmetric tricocycloid, where $v := c_1^{-1} c_0$ and γ is given by the action of $\mathfrak{S}_2$ on $\mathscr{O}(2)$.

Proof

So as to show the 3-cocycle ($(v \otimes 1)(1 \otimes \sigma)(v \otimes 1) = (1 \otimes v)(v \otimes 1)(1 \otimes v)$) and symmetry ($v(\gamma \otimes 1)v = (1 \otimes \gamma)v(1 \otimes \gamma)$) conditions of the tricocycloid, the proof consists in showing that the red paths in the diagrams that follow coincide, and then use the monicity hypothesis in the 3-cocycle one to get the desired result (in the symmetry one, we get the result because it is postcomposed by two invertible maps). Note that the tree on each node depicts the path from this node to the center of the diagram.

A fortunate phenomenon happens, enabling us to bypass the tedious computations that a more direct approach would have required! By postcomposing by the two red monic maps going to the center of the diagrams, the proof reduces to exploiting the (elementary) commutativity of the inner diagrams, step by step (as indicated by the colored numbers and highlightings: starting from the silver 0 all the way to the yellow 5).





■

Let $\mathbb{B}$ be an effectus. As $\mathbb{B}^{\mathrm{op}}$ is cartesian, we can define the following suboperad of End_1 in $\mathbb{B}^{\mathrm{op}}$ (note that we get rid of the maps factorising through a strictly smaller number of summands to overcome the fact the maps $\sigma\colon 1 \to X + 1$ that can be normalised in an effectus with normalisation are subject to a 'non-zero' condition ($\sigma \neq \kappa_2$)):

$$\begin{aligned} \mathscr{O}(k) \;&:=\; \mathrm{Hom}_{\mathbb{B}^{\mathrm{op}}}\,(1 \times \cdots \times 1, 1) \backslash \{\text{maps that factor through } \overbrace{1 \times \cdots \times 1}^{i<k \text{ factors}}\} \\ &= \mathrm{Hom}_{\mathbb{B}}\,(1, 1 + \cdots + 1) \backslash \{\text{maps that factor through } \underbrace{1 + \cdots + 1}_{i<k \text{ summands}}\} \end{aligned}$$

As mentioned earlier, we will now apply Lemma 3.1 to this operad, when $\mathbb{B}$ has normalisation. We can do so (the hypotheses are satisfied) because of:

Lemma 3.2 If $\mathbb{B}$ has normalisation, for all $k \geq 2$, the maps

$$c_k \;:=\; c_{1,k}(-,1,-)\colon \begin{cases} \mathrm{Hom}_{\mathbb{B}}\,(1,2) \times \mathrm{Hom}_{\mathbb{B}}\,(1,k) & \longrightarrow \mathrm{Hom}_{\mathbb{B}}\,(1,k+1) \\ 1 \overset{r}{\mapsto} 2,\; 1 \overset{\tau}{\mapsto} k & \longmapsto 1 \overset{r}{\mapsto} 2 \xmapsto{1+\tau} 1+k \xmapsto{\quad} k+1 \end{cases}$$

are invertible.

Proof
We claim that c_k^{-1} sends $1 \overset{\rho}{\mapsto} k+1$ to the couple

$$1 \overset{\rho}{\mapsto} k+1 \xmapsto{\quad} 1+k \xmapsto{1+!} 2, \qquad \llparenthesis 1 \overset{\rho}{\mapsto} k+1 \xmapsto{\quad} k+1 \rrparenthesis$$

- $c_k^{-1} c_k = \mathrm{id}$: Let $1 \overset{\rho}{\mapsto} k+1 = c_k(x,y) = 1 \overset{y}{\mapsto} 2 \xmapsto{1+x} 1+k \xmapsto{\quad} k+1$.

Consider $\alpha := (\!|\mathsf{X}\rho|\!) = (\!|\mathsf{X}\mathsf{A}'(1+x)y|\!)$: we have

$$(\alpha+1)(!+1)\mathsf{X}\rho = \mathsf{X}\rho$$

But

$$\begin{aligned}
(!+1)\mathsf{X}\rho &= (!+1)\mathsf{X}\mathsf{A}'(1+x)y \\
&= (!+1)\times\mathsf{A}\mathsf{A}'(1+x)y && \text{as } \mathsf{X} = \times\mathsf{A} \\
&= \times(1+!)\mathsf{A}\mathsf{A}'(1+x)y \\
&= \times(1+!)(1+x)y \\
&= \times\underbrace{(1+!x)}_{=\mathrm{id}}y = \times y
\end{aligned}$$

The third and last lines yield $\times(1+!)\mathsf{A}\rho = \times(1+!)\mathsf{A}\mathsf{A}'(1+x)y = \times y$, and by uniqueness of normalisation, as $(x+1)(!+1)\mathsf{X}\rho = (x+1)\times y = \times(1+x)y = \mathsf{X}\rho$, we get $x = \alpha$

- $c_k c_k^{-1} = \mathrm{id}$:

$$c_k c_k^{-1}(\rho) = \mathsf{A}'(1+\underbrace{(\!|\mathsf{X}\rho|\!)}_{:=\alpha})(1+!)\mathsf{A}\rho$$

But, by normalisation: $(\alpha+1)(!+1)\mathsf{X}\rho = \mathsf{X}\rho = \times\mathsf{A}\rho$. Consequently:

$$\begin{aligned}
\times\mathsf{A}\rho &= (\alpha+1)(!+1)\mathsf{X}\rho \\
&= (\alpha+1)\times(1+!)\mathsf{A}\rho \\
&= \times(\alpha+1)(1+!)\mathsf{A}\rho \\
&= \times\mathsf{A}'(\alpha+1)(1+!)\mathsf{A}\rho = \times c_k c_k^{-1}(\rho)
\end{aligned}$$

which yields the result.

■

With Lemma 3.2, it is not hard to show that the hypotheses of Lemma 3.1 are satisfied for the operad $\mathscr{O}$ we have defined in case $\mathbb{B}$ has normalisation, a corollary of which is

Corollary — Scalars of an effectus with normalisation form a tricocycloid. If $\mathbb{B}$ is an effectus with normalisation,

$$H := \mathrm{Hom}_{\mathbb{B}}(1,2)\backslash\{\kappa_1,\kappa_2\}$$

is a symmetric tricocycloid, where γ is postcomposition by $\times$, and v sends $r, s\colon 1 \to 2$ to

$$v(r,s) := \underbrace{1 \overset{r}{\mapsto} 2 \overset{s+1}{\longmapsto}\overset{\mathsf{A}}{\longmapsto} 1+2 \overset{1+!}{\longmapsto} 2}_{=\,r\cdot s \text{ in the effect monoid } M_{\mathbb{B}}}, \quad (\!|1 \overset{r}{\mapsto} 2 \overset{s+1}{\longmapsto}\overset{\mathsf{X}}{\longmapsto} 2+1|\!)$$

Theorem — Scalars of an effectus with normalisation form a left/double-cancellative tricocycloid (??). If $\mathbb{B}$ is an effectus with normalisation, $H := \mathrm{Hom}_{\mathbb{B}}(1,2)\backslash\{\kappa_1,\kappa_2\}$ is a left/double-cancellative tricocycloid.

We have already shown that H is a tricocycloid in Set. On top of that, $M_{\mathbb{B}} = H \cup \{\kappa_1,\kappa_2\}$ forms an effect monoid with normalisation, as seen in Theorem G.1. Thus, by Corollary D.1, $\cdot$ is left-cancellative away from $0 := \kappa_2 \in H \cup \{\kappa_1,\kappa_2\}$, and by distributivity, H satisfies the double cancellation property.

We now prove a result relating an effectus to the Kleisli category of the distribution monad over its scalars, thereby shedding light on one of the introductory questions.

3.4 Effectus and Kleisli category of the distribution monad over the scalars

Definition 3.3 — Pairwise orthogonality: n predicates $p_1, \ldots, p_n \colon X \to 2$ are said to be pairwise orthogonal if there exists a common bound $b \colon X \to n+1$ such that $\forall 1 \leq i \leq n, \quad [\rhd_i, \kappa_2]\, b = p_i$. As it happens, their sum is defined as $\bigovee_{i=1}^{n} p_i \quad := \quad (\nabla + \mathrm{id})\, b$.

NB such a bound b is unique by joint monicity of the family $([\rhd_i, \kappa_2])_{1 \leq i \leq n}$ (Lemma E.3)

Proposition — G.1 In $\mathbb{B}$, if n predicates $p_1, \ldots, p_n \colon X \to 2$ are pairwise orthogonal: for all $s_1, \ldots, s_n \colon 1 \to 2$, so are the predicates $p_1 \cdot s_1, \ldots, p_n \cdot s_n \colon X \to 2$.

NB Thanks to Proposition G.1, the sum $\bigovee_{i=1}^{n} r_i \cdot \phi_i(x)$ in the definition of the multiplication of $\mathcal{D}_M$ (1.1) is well defined.

Theorem — G.2. Let $\mathbb{B}$ be an effectus whose objects are finite coproducts of 1. Then

$$\mathbb{B} \cong \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})$$

Proof
In what follows, as $\mathrm{Hom}_{\mathbb{B}}(m, n) \cong \prod_{1 \leq i \leq m} \mathrm{Hom}_{\mathbb{B}}(1, n)$ by universal property of the coproduct, we denote by $q^1, \ldots, q^m \colon 1 \to n$ the m morphisms associated by this isomorphism to a morphism $q = [q^1, \ldots, q^m] \colon m \to n$.

We put

$$F \; := \; \begin{cases} \mathbb{B} & \longrightarrow \; \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}}) \\ n & \longmapsto \; n \\ m \xrightarrow{q} n & \longmapsto \; \begin{cases} m & \longrightarrow \; \mathcal{D}_{M_{\mathbb{B}}}(n) \\ k & \longmapsto \; \displaystyle\sum_{1 \leq i \leq n} \rhd_i \, q^k \, |i\rangle \end{cases} \end{cases}$$

In Theorem G.2, we show that F is a fully faithful functor, *i.e.* – as it is clearly bijective-on-objects – an isomorphism. ■

4. Additional results and new prospects

Due to lack of space, we briefly lay out some of the results that can be proven using the previous lemmas:

- When M has normalisation:
  - objects in Conv_M are given by binary convex sums, which generalises the situation observed by Garner in [Gar18].
  - $\mathcal{D}_{M_\mathbb{B}}$-algebras coincide with idempotent commutative $\star_H$-monoids, which implies that $\mathcal{D}_{M_\mathbb{B}}$ is linear exponential. In comparison, Jacobs shows in [JWW15] that, in the special case where $M_\mathbb{B} = [0,1]$, Stat preserves $+$ (thereby becoming a map of effectuses, which enable us to see the state-and-effect triangle associated to the effectus as a triangle in the category of effectuses). It is not completely clear yet if we can generalise this to the situation where $M_\mathbb{B}$ is an effect monoid with normalisation, but the result is likely to hold as well.

- Lawvere theory-related results (for a brief survey about Lawvere theories, see [Bár13; Gar13; HP07]):
  - As $\mathcal{D}_{M_\mathbb{B}}$ is a finitary monad (*i.e.* preserving filtered colimits): its category of $\mathcal{D}_{M_\mathbb{B}}$-algebras is equivalent to the category of models of its associated Lawvere theory ${\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(\mathcal{D}_{M_\mathbb{B}})$, *i.e.* the category of finite product preserving functors from ${\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(\mathcal{D}_{M_\mathbb{B}})$ to **Set**:

$$\mathcal{EM}(\mathcal{D}_{M_\mathbb{B}}) \simeq \mathrm{FProd}\,({\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(\mathcal{D}_{M_\mathbb{B}}), \mathbf{Set})$$

    and we hence have the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{EM}(\mathcal{D}_{M_\mathbb{B}}) \simeq \mathrm{FProd}\,({\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(\mathcal{D}_{M_\mathbb{B}}), \mathbf{Set}) \arrow[d, hook] \\
\mathcal{K}\ell_\mathbb{N}(\mathcal{D}_{M_\mathbb{B}}) \arrow[r, hook, "i"] & \mathbb{B} \arrow[ur, dotted, "\mathrm{Stat}"] \arrow[r, "\mathrm{N}_i := \mathrm{Hom}_\mathbb{B}(i(=),-)"'] & {[{\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(\mathcal{D}_{M_\mathbb{B}}), \mathbf{Set}]}
\end{tikzcd}$$

    It then appears that Stat is fully-faithful $\iff \mathrm{N}_i$ is fully-faithful $\iff i$ is dense (see [Kad] for a student's elmentary exposition on the nerve functor).
  - Since the adjunction $T_{-} \dashv {\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(-)$ – with [HP07]'s notations, where T_{-} is the functor sending a Lawvere theory to its associated monad – restricts to an equivalence between the category of Lawvere theories and the full replete reflective subcategory of finitary monads over **Set**:

$${\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(\mathcal{D}_{M_\mathbb{B}}) \cong {\mathcal{K}\ell_\mathbb{N}}^{\mathrm{op}}(T_{\mathbb{B}^{\mathrm{op}}}) \Longrightarrow \mathcal{D}_{M_\mathbb{B}} \cong T_{\mathbb{B}^{\mathrm{op}}}$$

  - This explains where Stat comes from in a natural way, to some extent.

- Models of ILL: if M has normalisation, $\mathcal{D}_M$ is a linear exponential monad: does it live in a model of linear logic? We sketch some arguments in this respect:
  1. *First attempt*: $(\mathrm{Set}, \star_{\mathring{M}})$ is symmetric monoidal, but since the initial object is a unit for $\star_{\mathring{M}}$, it cannot be closed. Despite this, we may still wonder if we have a model of MELL^{-} (without units). This remains of significant interest, since Girard's proof nets do not incorporate units, and attempts to take them into account can be argued to be unsatisfactory to some extent, in that they are considerably more intricate and fail at providing purely geometric proof normal forms. In this sense, equivalence of MLL^{-} proofs have been very soon shown to be in **PTIME** via Girard's proof nets, while an analogous result for MLL remains an open problem.
     In his PhD thesis [Hou13], Houston shows that the ordinary definition of a SMCC where every mention of the unit object is removed (which he calls a 'unitless SMCC') is *not enough* to have a

model of MLL^-. But he gives a simple criterion fixing this, *viz.* that every arrow $A \to B$ stems from a unique *linear element* of $A \multimap B$ (*i.e.* a natural transformation $\gamma_X : X \to (A \multimap B) \otimes X$ such that $\alpha_{A \multimap B, X, Y}(\gamma_X \otimes Y) = \gamma_{X \otimes Y} : X \otimes Y \to (A \multimap B) \otimes (X \otimes Y)$). Unfortunately, this criterion is not satisfied in our case.

2. To circumvent the '0 being the unit for the tensor' problem, we may go about considering $\star_M$, where M is seen as a lax tricocycloid. But in this case, Ross Street's construction only yields a lax semi-monoidal category.
3. If a functor $F : \mathbf{A} \to \mathbf{B}$ is lax monoidal [nLaa], then the left Kan extension $\mathrm{Lan}_F : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ can be shown to be lax monoidal. So if $T : \mathbf{A} \to \mathbf{B}$ is an opmonoidal monad, $T^{\mathrm{op}} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$ is lax monoidal, and $\mathrm{Lan}_F : [\mathbf{A}, \mathbf{Set}] \to [\mathbf{B}, \mathbf{Set}]$ is lax monoidal, thus ${\mathrm{Lan}_F}^{\mathrm{op}} : [\mathbf{A}, \mathbf{Set}]^{\mathrm{op}} \to [\mathbf{B}, \mathbf{Set}]^{\mathrm{op}}$ is oplax monoidal. But if T is linear exponential, does ${\mathrm{Lan}_F}^{\mathrm{op}}$ remain linear exponential? We believe so, but have not proven it.
4. In the same vein as the previous approach, we may also try to Kan extend $\mathbf{y}_{\mathbf{Set}}\, \mathcal{D}_M$ along the Yoneda embedding instead.

Lastly, we did not have time to delve into this, but it might be of interest to try to link what we did with Girard's quantum coherence spaces [Gir03]. These do not model the exponential modalities, because Girard only tackles the finite-dimensional case, but we may have a take on the matter with the generalised distribution monad being linear exponential.

When it comes to the *3-cocycle* condition satisfied by a tricocycloid, it comes from non-abelian cohomology, and corresponds to a form of higher-dimensional coherence condition (in our case: associativity) in higher categories. A tricocycloid corresponds to an element of dimension 4 in the nerve of the tricategory with one object and one morphism that constitutes the double delooping of the braided monoidal category in which the tricocycloid is defined. For more details about this, see the seminal paper of Ross Street, which investigates and clarifies this link between higher-dimensional coherence conditions and non-abelian cohomology: [Str87].

4.1 Conclusion

Throughout this internship, we have related effect monoids, tricocycloids and effectuses when they satisfy some conditions reminiscent of a form of generalised 'normalisation' (of probability subdistributions). Our notions of effect monoids with normalisation and tricycloids with left/double cancellation are directly related to Jacobs' notion of effectus with normalisation. Given an effectus with normalisation, its effect monoid of scalars has normalisation, and if we remove the scalars 0 and 1 (corresponding to the two coprojections κ_1, κ_2), it also forms a symmetric tricocycloid with left/double cancellation. On top of that, we have that

- the categories of effect monoids with normalisation and of symmetric left/double cancellative tricocycloids in Set are isomorphic
- when the effect monoid M has normalisation, the convex spaces over M are given by abstract binary sums, the $\mathcal{D}_M$-algebras are idempotent commutative monoids for the tensor induced by the tricocycloid associated to M, and $\mathcal{D}_M$ is linear exponential
- we have exhibited the generalised distribution monad $\mathcal{D}_{M_{\mathbb{B}}}$ over the scalars of an effectus whose objects are finite coproducts of the terminal object 1 as the monad associated to the Lawvere theory $\mathbb{B}^{\mathrm{op}}$, due to $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})$ and $\mathbb{B}$ being isomorphic. In the general case, we have seen that $\mathbb{B}$ can be embedded, under some equivalent conditions (among which the fully-faithfulness of the state functor), in the category of presheaves over $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})^{\mathrm{op}}$, which gives a natural explanation of the 'origin' of the state functor.
- we have sketched some ideas to come up with models of linear logic based on the generalised distribution monad $\mathcal{D}_M$.

We now have a set of results that can serve as a toolbox to further investigate the relationship between various fields, via the triad: effect monoids (quantum mechanics/logic), tricocycloids (quantum algebra), and effectuses (quantum and probabilistic logic and computation). We refer the reader to the first 'overview' section for more details about future prospects.

Bibliography

Front cover background image: Jamie Farrant / Getty Images

Articles

[BD97] John C. Baez and James Dolan. ‘Higher-Dimensional Algebra III: N-Categories and the Algebra of Opetopes’. In: (10th Feb. 1997). arXiv: `q-alg/9702014`. URL: `http://arxiv.org/abs/q-alg/9702014` (visited on 30/06/2019).

[BV36] Garrett Birkhoff and John Von Neumann. ‘The Logic of Quantum Mechanics’. In: *Annals of Mathematics* 37.4 (1936), pages 823–843. ISSN: 0003-486X. DOI: `10.2307/1968621`. URL: `https://www.jstor.org/stable/1968621` (visited on 29/06/2019).

[CL] Eugenia Cheng and Aaron Lauda. ‘Higher-Dimensional Categories:’ in: (), page 182.

[Cho+15] Kenta Cho et al. ‘An Introduction to Effectus Theory’. In: (17th Dec. 2015). arXiv: `1512.05813 [quant-ph]`. URL: `http://arxiv.org/abs/1512.05813` (visited on 06/08/2019).

[De 14] Valeria De Paiva. ‘Categorical Semantics of Linear Logic For All’. In: (2nd Mar. 2014). DOI: `10.1007/978-94-007-7548-0_9`.

[FB94] D. J. Foulis and M. K. Bennett. ‘Effect Algebras and Unsharp Quantum Logics’. In: *Foundations of Physics* 24.10 (1st Oct. 1994), pages 1331–1352. ISSN: 1572-9516. DOI: `10.1007/BF02283036`. URL: `https://doi.org/10.1007/BF02283036` (visited on 02/08/2019).

[Gar13] Richard Garner. ‘Lawvere Theories, Finitary Monads and Cauchy-Completion’. In: (10th July 2013). arXiv: `1307.2963 [math]`. URL: `http://arxiv.org/abs/1307.2963` (visited on 06/08/2019).

[Gar18] Richard Garner. ‘Hypernormalisation, Linear Exponential Monads and the Giry Tricocycloid’. In: (6th Nov. 2018). arXiv: `1811.02710 [cs, math]`. URL: `http://arxiv.org/abs/1811.02710` (visited on 29/04/2019).

[HZ08] Teiko Heinosaari and Mario Ziman. ‘Guide to Mathematical Concepts of Quantum Theory’. In: *Acta Physica Slovaca. Reviews and Tutorials* 58.4 (1st Aug. 2008). ISSN: 1336-040X, 0323-0465. DOI: `10.2478/v10155-010-0091-y`. arXiv: `0810.3536`. URL: `http://arxiv.org/abs/0810.3536` (visited on 13/05/2019).

[Hou13] Robin Houston. ‘Linear Logic without Units’. In: (9th May 2013). arXiv: `1305.2231 [cs, math]`. URL: `http://arxiv.org/abs/1305.2231` (visited on 06/08/2019).

[HP07] Martin Hyland and John Power. ‘The Category Theoretic Understanding of Universal Algebra: Lawvere Theories and Monads’. In: *Electronic Notes in Theoretical Computer Science*. Computation, Meaning, and Logic: Articles Dedicated to Gordon Plotkin 172 (1st Apr. 2007), pages 437–458. ISSN: 1571-0661. DOI: `10.1016/j.entcs.2007.02.019`. URL: `http://www.sciencedirect.com/science/article/pii/S1571066107000874` (visited on 06/08/2019).

[Jac11] Bart Jacobs. ‘Probabilities, Distribution Monads, and Convex Categories’. In: *Theoretical Computer Science*. Festschrift in Honour of Jan Bergstra 412.28 (20th June 2011), pages 3323–3336. ISSN: 0304-3975. DOI: `10.1016/j.tcs.2011.04.005`. URL: `http://www.sciencedirect.com/science/article/pii/S0304397511002817` (visited on 29/04/2019).

[Jac15] Bart Jacobs. 'New Directions in Categorical Logic, for Classical, Probabilistic and Quantum Logic'. In: *Logical Methods in Computer Science* 11.3 (1st Oct. 2015), page 24. ISSN: 18605974. DOI: 10.2168/LMCS-11(3:24)2015. arXiv: 1205.3940. URL: http://arxiv.org/abs/1205.3940 (visited on 13/05/2019).

[Jac17a] Bart Jacobs. 'A Recipe for State-and-Effect Triangles'. In: *Logical Methods in Computer Science ; Volume 13* (2017), Issue 2, 18605974. ISSN: 1860-5974. DOI: 10.23638/LMCS-13(2:6)2017. arXiv: 1703.09034. URL: http://arxiv.org/abs/1703.09034 (visited on 13/05/2019).

[Jac17b] Bart Jacobs. 'Hyper Normalisation and Conditioning for Discrete Probability Distributions'. In: *Logical Methods in Computer Science ; Volume 13* (2017), Issue 3, 18605974. ISSN: 1860-5974. DOI: 10.23638/LMCS-13(3:17)2017. arXiv: 1607.02790. URL: http://arxiv.org/abs/1607.02790 (visited on 19/08/2019).

[Jac18] Bart Jacobs. 'From Probability Monads to Commutative Effectuses'. In: *Journal of Logical and Algebraic Methods in Programming* 94 (1st Jan. 2018), pages 200–237. ISSN: 2352-2208. DOI: 10.1016/j.jlamp.2016.11.006. URL: http://www.sciencedirect.com/science/article/pii/S2352220816301122 (visited on 29/04/2019).

[JM12] Bart Jacobs and Jorik Mandemaker. 'Coreflections in Algebraic Quantum Logic'. In: *Foundations of Physics* 42.7 (1st July 2012), pages 932–958. ISSN: 1572-9516. DOI: 10.1007/s10701-012-9654-8. URL: https://doi.org/10.1007/s10701-012-9654-8 (visited on 20/08/2019).

[Jim89] Michio Jimbo. 'Introduction to the Yang-Baxter Equation'. In: *International Journal of Modern Physics A* 04.15 (1st Sept. 1989), pages 3759–3777. ISSN: 0217-751X. DOI: 10.1142/S0217751X89001503. URL: https://www.worldscientific.com/doi/abs/10.1142/S0217751X89001503 (visited on 29/04/2019).

[Kad] Younesse Kaddar. 'Event Structures as Presheaves'. In: (), page 57.

[Koc71] Anders Kock. 'Bilinearity and Cartesian Closed Monads.' In: *MATHEMATICA SCANDINAVICA* 29 (1st Dec. 1971), page 161. ISSN: 1903-1807, 0025-5521. DOI: 10.7146/math.scand.a-11042. URL: http://www.mscand.dk/article/view/11042 (visited on 20/08/2019).

[Kup] Greg Kuperberg. 'A Concise Introduction to Quantum Probability, Quantum Mechanics, and Quantum Computation'. In: (), page 36.

[Lan63] Saunders Mac Lane. 'Natural Associativity and Commutativity'. In: *Rice University Studies* (1963).

[Lei16] Tom Leinster. 'Basic Category Theory'. In: (29th Dec. 2016). arXiv: 1612.09375 [math]. URL: http://arxiv.org/abs/1612.09375 (visited on 23/06/2019).

[Maj06] Shahn Majid. 'A Quantum Group?' In: *Notices of the AMS* 53.1 (2006).

[Mel] Paul-André Melliès. 'Categorical Semantics of Linear Logic'. In: (), page 213.

[MG99] T. Moreland and S. Gudder. 'Infima of Hilbert Space Effects'. In: *Linear Algebra and its Applications* 286.1 (1st Jan. 1999), pages 1–17. ISSN: 0024-3795. DOI: 10.1016/S0024-3795(98)10119-2. URL: http://www.sciencedirect.com/science/article/pii/S0024379598101192 (visited on 21/08/2019).

[Sch] Andrea Schalk. 'What Is a Categorical Model for Linear Logic?' In: (), page 26.

[Seg47] I. E. Segal. 'Irreducible Representations of Operator Algebras'. In: *Bulletin of the American Mathematical Society* 53.2 (1st Feb. 1947), pages 73–89. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1947-08742-5. URL: http://www.ams.org/journal-getitem?pii=S0002-9904-1947-08742-5 (visited on 12/08/2019).

[Sta15] Allen Stairs. 'Quantum Logic and Quantum Reconstruction'. In: *Foundations of Physics* 45.10 (Oct. 2015), pages 1351–1361. ISSN: 0015-9018, 1572-9516. DOI: 10.1007/s10701-015-9879-4. arXiv: 1501.05492. URL: http://arxiv.org/abs/1501.05492 (visited on 14/05/2019).

[Sta] Jim Stasheff. 'What Is an Operad'. In: *Notices of the AMS* 51.6 ().

[Str87] Ross Street. 'The Algebra of Oriented Simplexes'. In: *Journal of Pure and Applied Algebra* 49.3 (1st Dec. 1987), pages 283–335. ISSN: 0022-4049. DOI: 10.1016/0022-4049(87)90137-X. URL: http://www.sciencedirect.com/science/article/pii/002240498790137X (visited on 24/06/2019).

[Str98] Ross Street. 'Fusion Operators and Cocycloids in Monoidal Categories'. In: *Applied Categorical Structures* 6.2 (1st June 1998), pages 177–191. ISSN: 1572-9095. DOI: 10.1023/A:1008655911796. URL: https://doi.org/10.1023/A:1008655911796 (visited on 29/04/2019).

[Zap] Octavio Zapata. 'Effectus of Quantum Probability on Relational Structures'. In: (), page 12.

Books

[Arv98] William Arveson. *An Invitation to C*-Algebras*. Corr. 2nd print. Graduate Texts in Mathematics 39. New York: Springer, 1998. 106 pages. ISBN: 978-0-387-90176-3.

[DM05] Lokenath Debnath and Piotr Mikusiński. *Hilbert Spaces with Applications*. OCLC: 288957631. Amsterdam; Boston: Elsevier Academic Press, 2005.

[Dix77] Jacques Dixmier. *C*-algebras*. North-Holland mathematical library v. 15. Amsterdam ; New York : New york: North-Holland ; sole distributors for the U.S.A. and Canada, Elsevier North Holland, 1977. 492 pages. ISBN: 978-0-7204-0762-4.

[Fre09] Benoit Fresse. *Modules over Operads and Functors*. Lecture Notes in Mathematics 1967. OCLC: ocn269435377. Berlin: Springer, 2009. 308 pages. ISBN: 978-3-540-89055-3 978-3-540-89056-0.

[Lan78] Saunders Mac Lane. *Categories for the Working Mathematician*. 2nd edition. Graduate Texts in Mathematics. New York: Springer-Verlag, 1978. ISBN: 978-0-387-98403-2. URL: https://www.springer.com/gp/book/9780387984032 (visited on 23/06/2019).

[May89] James P. May. *The Geometry of Iterated Loop Spaces*. 1. ed., 2. print. Lecture Notes in Mathematics 271. OCLC: 258500276. Berlin: Springer, 1989. 175 pages. ISBN: 978-0-387-05904-4 978-3-540-05904-2.

[NW18] John von Neumann and Nicholas A Wheeler. *Mathematical Foundations of Quantum Mechanics*. OCLC: 1021172445. 2018. ISBN: 978-1-4008-8992-1 978-0-691-17857-8 978-0-691-17856-1. URL: http://search.ebscohost.com/login.aspx?direct=true&scope=site&db=nlebk&db=nlabk&AN=1629183 (visited on 30/06/2019).

[Rie17] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017. ISBN: 978-0-486-82080-4. URL: https://books.google.com.au/books?id=6B9MDgAAQBAJ.

[Sak98] Shôichirô Sakai. *C*-Algebras and W*-Algebras*. Classics in Mathematics. Berlin ; New York: Springer, 1998. 256 pages. ISBN: 978-3-540-63633-5.

[TT03] Masamichi Takesaki and Masamichi Takesaki. *Theory of Operator Algebras*. Encyclopaedia of Mathematical Sciences, Operator Algebras and Non-Commutative Geometry v. 124-125, 127 5-6, 8. OCLC: ocm51395972. Berlin ; New York: Springer, 2003. 2 pages. ISBN: 978-3-540-42248-8 978-3-540-42914-2 978-3-540-42913-5.

Online

[nLaa] nLab authors. *Lax Monoidal Category in nLab*. URL: https://ncatlab.org/nlab/show/lax+monoidal+category (visited on 22/08/2019).

[nLab] nLab authors. *Module over a Monoid in nLab*. URL: https://ncatlab.org/nlab/show/module+over+a+monoid (visited on 20/08/2019).

Appendix

A. Notations

General notations

$\cong$ Isomorphism	$\simeq$ Equivalence
$gf = g \circ f = f;g$ if $f\colon X \to Y$, $g\colon Y \to Z$: morphism composition	$\mathrm{id}\colon X \to X$ or $1\colon X \to X$ Identity morphism
$\kappa_i \quad (i \in \mathbb{N})$ Coprojections (if a coproduct is involved)	$\pi_i \quad (i \in \mathbb{N})$ Projections (if a product is involved)
$[f, g]\colon A + B \to X$ copairing of $f\colon A \to X$ and $g\colon B \to X$	$f + g\colon A + B \to X + Y$ coproduct $[\kappa_1 f, \kappa_2 g]$ of $f\colon A \to X$ and $g\colon B \to Y$
$\mathbb{0}$, $\mathbb{1}$ or 0, 1 initial and terminal objects	$!\colon \mathbb{0} \to X$, $!\colon X \to \mathbb{1}$ initial/terminal morphism (when not the exponential modality in linear logic)
$n \in \mathscr{C}$ in a category $\mathscr{C}$ with terminal object and coproducts: $(1 + \cdots + 1) + 1$ (n times)	$n \cdot X$ $X + \cdots + X$ (n times)
$\rhd_i\colon X_1 + \cdots + X_n \to X_i + 1$ $[\kappa_2!, \dots, \kappa_2!, \kappa_1, \kappa_2!, \dots, \kappa_2!]$ where κ_1 is in position i, everywhere else: $\kappa_2\,!$	$f\colon A \rightharpoonup B$ Partial map from A to B
$\nabla\colon X + \cdots + X \to X$ codiagonal $[\mathrm{id}, \dots, \mathrm{id}]\colon X + \cdots + X \to X$	**Set** category of sets and functions
$\mathcal{K}\ell(\mathbb{T})$ Kleisli category of the monad $\mathbb{T}$	$\mathcal{EM}(\mathbb{T})$ Eilenberg-Moore category of $\mathbb{T}$
$\mathcal{K}\ell_{\mathbb{N}}(\mathbb{T})$ Full subcategory of $\mathcal{K}\ell(\mathbb{T})$ with finite coproducts of 1 ("numbers" n) as objects	
$\mathcal{B}(\mathscr{H}) \subseteq \mathrm{Hom}_{\mathbf{Hilb}}(\mathscr{H}, \mathscr{H})$ Bounded linear maps on a Hilbert space $\mathscr{H}$	$\mathcal{DM}(\mathscr{H})$ Set of density matrices of a finite-dimensional Hilbert space $\mathscr{H}$ Non-standard notations will be introduced when used for the first time, but for convenience, a glossary of notations can be found in appendix A.

If $\mathbb{B}$ is an effectus (definition 3.1), we denote by

- $\mathrm{Pred} := \mathrm{Hom}_{\mathbb{B}}(-, 1+1)$ its predicate functor
- $\mathrm{Stat} := \mathrm{Hom}_{\mathbb{B}}(1, -)$ its state functor
- $M_{\mathbb{B}} := \mathrm{Pred}(1) = \mathrm{Stat}(2) = \mathrm{Hom}_{\mathbb{B}}(1, 2)$ its effect monoid of scalars (Theorem E.6)

Handy "shuffle maps" notations (the symbols illustrate what the maps do)

$:= [[\kappa_1, \kappa_2\kappa_1], \kappa_2\kappa_2]\colon (X+Y)+Z \to X+(Y+Z)$

$:= {}^{-1}\colon X+(Y+Z) \to (X+Y)+Z$

$:= [[\kappa_1, \kappa_2!], \kappa_2]\colon (X+X)+1 \to X+1$

$:= [[\kappa_2!, \kappa_1], \kappa_2]\colon (X+X)+1 \to X+1$

$:= (X+Y)+Z \longrightarrow X+(Y+Z) \longrightarrow (Y+Z)+X$

$:= {}^{-1}\colon (Y+Z)+X \to (X+Y)+Z$

$:= [\kappa_2, \kappa_1]\colon X+Y \to Y+X$

$:= [[\kappa_1, \kappa_3], \kappa_2]\colon (X+Y)+Z \to (X+Y)+Z$

$:= [[\kappa_2, \kappa_1], \kappa_3]\colon (X+Y)+Z \to (X+Y)+Z$

A.0.1 Abbreviations

Table A.1: Abbreviations/Acronyms

General	**iff**: if and only if **resp.**: respectively **wlog**: without loss of generality **cf.**: see **v/vs/vs.**: versus
Effect Algebras	**PCM**: partial commutative monoid **EA**: effect algebra
Categories	**SMC**: symmetric monoidal category **SMCC**: symmetric monoidal closed category
Linear Logic	**ILL/CLL**: intuitionistic/classical linear logic **MLL**: multiplicative linear logic **MALL**: multiplicative additive linear logic **MELL**: multiplicative exponential linear logic

B. Effect algebras

An effect algebra carries a poset structure by setting $x \leq y \iff \exists z; \quad x \ovee z = y$. A *partial difference* can also be given by $y \ominus x = z \iff x \ovee z = y$. Moreover, in an effect algebra, the sum is easily shown to be left-cancellative, since $a \ovee b = a \ovee b' \implies b = (a \ovee (a \ovee b)^{\perp})^{\perp} = (a \ovee (a \ovee b')^{\perp})^{\perp} = b'$.

Example B.1 — Non-examples of effect algebras
Here are two key example of PCMs that are *not* effect algebras.

Non-examples
The set of partial functions from a set X to $[0,1]$ can be endowed with two PCM structures, by setting 0 to be the empty function (with empty domain), and

- $f \ovee g$ to be the copairing (disjoint union) of f and $g\colon X \to [0,1]$ when $f \perp g \iff \operatorname{dom} f \cap \operatorname{dom} g = \emptyset$
- $f \ovee g$ to be the pointwise sum of f and $g\colon X \to [0,1]$ when $f \perp g \iff \forall x \in \operatorname{dom} f \cap \operatorname{dom} g.\, f(x) + g(x) \leq 1$

But in both cases, if we set 1 to be the only reasonable choice (the constant function equal to 1), we do not have an effect algebra: in the first (resp. second) case, the existence (resp. uniqueness) of the orthocomplement does not always hold; indeed, consider a partial function such that $\operatorname{dom} f = \{0\}$ and $f(0) < 1$: it has no orthocomplement under copairing (resp. $f(0) = 1$, in which case the two partial functions which are equal to 1 everywhere except at 0, where one is not defined and the other equals 0, are both orthocomplement). Note that these examples can easily be generalised to the set of functions from a set X to an arbitrary effect monoid M.

Example
In the previous non-example: the second case (pointwise addition) can be made into an effect algebra by restraining the codomain of the partial functions to be $(0,1]$ (excluding 0), or $M \backslash \{0,1\}$ in the general case.

C. Category theory reminders

C.1 Monoids, monads, and modules

Monoids are pervasive in category theory. The most basic instances that spring to mind are monoids in the category Set of sets, which will henceforth be referred to as ”classical” monoids. This section is a quick and dirty reminder of related notions that will come in handy later, but we refer to [Lan78], [Rie17], and [Lei16] for more information on the categorical background.

Roughly, and at the risk of oversimplifying, in the toolbox of a seasoned category theorist, there are three main ways to generalise a given mathematical structure – let's say a monoid M in the "classical" sense (*i.e.* in Set), for example – : they may either resort to

1. a process known as *vertical categorification*, also called **enrichment**: as the underlying structure of our monoid M is a set (seen as a discrete category), two elements $x, y \in M$ are either *equal* or *not equal*. This can be thought of as: either the set of morphisms between x, y is $\mathrm{Hom}_M(x,y) = \{*\} \overset{\text{written as}}{=} 1$ (when $x = y$), or $\mathrm{Hom}_M(x,y) = \emptyset \overset{\text{written as}}{=} 0$ ($x \neq y$). As it happens, for every $x, y \in M$, $\mathrm{Hom}_M(x,y)$ is an object of the category[1] $\mathbb{2} := 0 \xrightarrow{\star} 1$: our discrete category M is said to be **enriched** over $\mathbb{2}$ (and more generally, the same goes for every preorder: the preorder is trivial in a set). Vertical categorification consists in replacing $\mathbb{2}$ by any other category $\mathscr{C}$ (under a few assumptions so that it makes sense to do so, *viz.* $\mathscr{C}$ is monoidal). Most of the times, this involves replacing equalities by coherent isomorphisms/equivalences.
2. or *horizontal categorification*, also called **oidification**: the ultimate example of oidification is going from a group to a *groupoid*, or, to retrieve our previous example, from a monoid to a category: a monoid is easily seen to be nothing but a category with exactly one-object (the morphisms correspond to the element of the monoid, categorical composition being the monoid multiplication and the identity morphism the unit element). Horizontal categorification consists in getting rid of the "exactly one object" constraint: we arrive at the notion of categories[2], if we started with monoids (and of groupoids, from groups (one-object categories where all the morphisms are invertible)).
3. or a third possibility, called *internalisation*: our classical monoid (like many other traditional structures in mathematics) are defined in the category Set, in that they are nothing but a carrier set equipped with some extra-structure and satisfying certain properties[3]. Internalisation involves replacing Set with another "suitable" category $\mathscr{C}$ (preferably belonging to a class of categories containing Set): our formerly classical monoids then become monoids **internal to** $\mathscr{C}$ (also called **monoid objects** in $\mathscr{C}$, or simply *monoids* when the context is clear[4]). Oddly enough, what is meant by "suitable" is that $\mathscr{C}$ ought to be a (vertically) *categorified* version of the very same structure we are generalising from Set to $\mathscr{C}$! In our classical monoid example, $\mathscr{C}$ would have to be a vertically categorified monoid, *i.e.* a **monoidal category**. This is known as Baez's and Dolan's *microcosm principle*:

   > “[...] certain algebraic structures can be defined in any category equipped with a categorified version of the same structure. ([BD97, page 11])”

   In some cases, internalisation generalises vertical categorification. This is the case for enrichment over the category **Cat** of (small[5]) categories, for example: categories enriched over **Cat**, called **2-categories**, are categories internal to **Cat** (*double categories*) satisfying a special condition[6].

[1] which is, as required (to anticipate what is coming), a *monoidal* category (a 'categorified monoid'), with $\times$ as tensor and 1 as unit

[2] which should be called "monoidoids" if the terminology was respected analogously to groups and groupoids

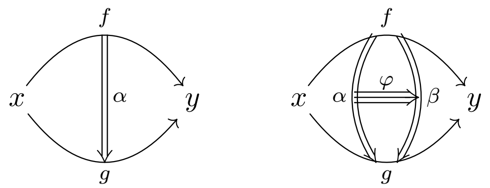
[3] incidentally, this is really the model-theoretic way to regard models of a given theory: the extra-structure is specified by the signature of the $\mathscr{L}$-structures, and the properties by the theory

[4] which will be the convention later, as monoids in the traditional sense are qualified as "classical"

[5] but for the sake of simplicity, size issues will be omitted in what follows

[6] *viz.* all the morphisms in the category of objects are trivial; in order to collapse the "cubical" 2-cells into "globular" ones, see [CL, section 5.1.3] for example.

2-categories, due to being enriched over **Cat**, can be thought of as 2-dimensional categories, hence paving the way for higher-dimensional category theory. By calling objects (resp. morphisms between objects) 0-cells/0-arrows/0-morphisms[7] (resp. 1-arrows between 0-arrows), there are 2-arrows between 1-arrows (corresponding to the morphisms of the hom-category). But we could go further: by enriching over the category 2-**Cat** of 2-categories, we get a 3-category: there are $n+1$-arrows between n-arrows for all $n \leq 2$.



x, y: 0-arrows
f, g: 1-arrows
α, β: 2-arrows
φ: 3-arrow

A 2-cell in a 2-category and a 3-cell in a 3-category

And by induction, enriching over the category n-**Cat** of n-categories for $n \in \mathbb{N}$, yields a $n+1$-category: to be more specific, a *strict* $n+1$-category. By *strict*, we mean that the associativity and unit laws of enriched categories hold *on the nose*; *e.g.* if f, g, h are 1-morphisms, the two 1-morphisms $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are *equal*: all the higher dimensional cells between them are identities. But in *weak* n-categories, the situation is more general in that these laws hold only up to (coherent) higher-dimensional isomorphisms: there are invertible $k+1$-arrows between the k-arrows which were made equal in strict n-categories, and these $k+1$-arrows satisfy themselves some coherence axioms. That being said, a **bicategory** (resp. **tricategory**) is a *weak* 2-category (resp. 3-category): therein, in the previous example, $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are not *a priori* equal, but there exists a 2-isomorphism $\alpha_{f,g,h} : (f \circ g) \circ h \Rightarrow f \circ (g \circ h)$ between them (satisfying a coherence law known as the *pentagon law*).

With these general principles in mind, we briefly recall a handful of definitions important for what follows.

Definition C.1 — A monoidal category $(\mathscr{C}, \otimes, I)$ is a category equipped with a bifunctor $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called **tensor product**, an object $I \in \mathscr{C}$ called **tensor unit** and structural natural isomorphisms: an associator $\alpha_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ satisfying the *pentagon law* ($(1 \otimes \alpha_{X,Y,Z})\alpha_{W,X \otimes Y,Z}(\alpha_{W,X,Y} \otimes 1) = \alpha_{W,X,Y \otimes Z}\, \alpha_{W \otimes X,Y,Z}$), a left unitor $\lambda_X \colon 1 \otimes X \to X$, and a right unitor $\rho_X \colon X \otimes 1 \to X$ satisfying the *triangle law* ($\rho_X \otimes 1 = (1 \otimes \lambda_Y)\alpha_{X,1,Y}$). $\mathscr{C}$ is said to be **strict** when the structural isomorphisms are identities.

In a monoidal category, an important consequence of the pentagon and triangle laws is Mac Lane's *coherence theorem for monoidal categories* [Lan63]: every diagram composed of structural morphisms commutes, which is equally to say that every monoidal category is (monoidally) equivalent to a strict one, and will enable us to assume that our monoidal categories are strict later.

Definition C.2 — A monoid (object) $M \in \mathscr{C}$ **in a monoidal category** $(\mathscr{C}, \otimes, I)$ is an object equipped with a multiplication $\mu \colon M \otimes M \to M$ which is associative ($\mu(\mu \otimes 1) = \mu(1 \otimes \mu)\alpha$) and a unit $\eta \colon M \to M$ satisfying the left ($\mu(\eta \otimes 1) = \lambda$) and right ($\mu(1 \otimes \eta) = \rho$) unit laws.

In a monoidal category, the tensor $\otimes$ being commutative is no longer a "boolean property", contrary to a classical monoid (where the operation is either commutative or not), as $\mathrm{Hom}_{\mathscr{C}}(X \otimes Y, Y \otimes X)$ may be any set, and not only either $\emptyset = 0 \in 2$ or $\{*\} = 1 \in 2$ anymore. As a matter of fact, more subtle ways in which $\otimes$ may be thought of as "commutative to a certain extent" are indicated by natural transformations $\sigma_{X,Y} \colon X \otimes Y \to Y \otimes X$ called "braidings" satisfying some assumptions.

Another germane notion is the concept of **delooping**

Example C.1 — Delooping examples
- monoid, group
- monoidal category
- braided monoidal category

[7] henceforth: "cell", "morphism", and "arrow" will be synonymous

D. Effect Monoids v Tricocycloids

D.1 Effect monoids with normalisation

Let M be an effect monoid.

Proposition D.1 For all $a, b \in M$, $a^\perp b^\perp = (a \ovee a^\perp b)^\perp = (b \ovee ab^\perp)^\perp$

Proof

$$1 = \begin{cases} a \ovee a^\perp = a \ovee a^\perp(b \ovee b^\perp) = a \ovee a^\perp b \ovee a^\perp b^\perp \\ b \ovee b^\perp = b \ovee (a \ovee a^\perp)b^\perp = b \ovee ab^\perp \ovee a^\perp b^\perp \end{cases}$$ ■

M is said to have **normalisation** iff $\forall a \neq 1, b \in M.\, a \perp b \implies \exists! c;\ b = a^\perp c$

Corollary D.1 If M has normalisation, M is *left-cancellative away from zero*, *i.e.*

$$\forall a \neq 0, b, b' \in M. \quad ab = ab' \quad \implies \quad b = b'$$

Proposition D.2 If M has normalisation, then for all $a \ovee b_1 \ovee \cdots \ovee b_n = 1$ with $a \neq 1$, there exist unique $c_1 \ovee \cdots \ovee c_n = 1$ such that $\forall i.\, b_i = a^\perp c_i$

Proof
By induction on n. The base case ($n = 1$) is obvious. Suppose the results holds for n, we prove it for $n + 1$. Given $a \ovee b_0 \ovee \cdots \ovee b_n = 1$, there exists, by normalisation, a unique c_0 such that $b_0 = a^\perp c_0$. Thus

$$(a \ovee a^\perp c_0) \ovee b_1 \cdots \ovee b_n = 1$$

and, by induction, we have unique elements $d_1 \ovee \cdots \ovee d_n = 1$ such that $b_i = (a \ovee a^\perp c_0)^\perp d_i \overset{\text{left cancellativity}}{=} a^\perp c_0^\perp d_i$ for all $1 \leq i \leq n$. So by taking $c_i = c_0^\perp d_i$ for all $1 \leq i \leq n$, we get $c_0 \ovee c_1 \ovee \cdots \ovee c_n = 1$ such that $b_i = a^\perp c_i$ for all $0 \leq i \leq n$. These c_i are unique, by left cancellativity. ■

D.2 Tricocycloids in Set

Lemma D.2 Let (H, v, γ) be a lax tricocycloid with a symmetry. It is a tricocycloid iff $v(\gamma \otimes \gamma)v(\gamma \otimes \gamma) = 1$, in which case $v^{-1} = (\gamma \otimes \gamma)v(\gamma \otimes \gamma)$.

Proof
If v is invertible, then it is easy to see that γ is a symmetry for v^{-1}, in that $v^{-1}(\gamma \otimes 1)v^{-1} = (1 \otimes \gamma)v^{-1}(1 \otimes \gamma)$, and

$$\begin{aligned}
&v \overbrace{(\gamma \otimes \gamma)}^{= (1\otimes\gamma)v^{-1}v(\gamma\otimes 1)} v(\gamma \otimes \gamma) \\
&\qquad = v(1 \otimes \gamma)v^{-1}v(\gamma \otimes 1)v(\gamma \otimes \gamma) \\
&\qquad = v(1 \otimes \gamma)v^{-1}(1 \otimes \gamma)v(1 \otimes \gamma)(\gamma \otimes \gamma) && \text{(by due to } \gamma \text{ being a symmetry for } v) \\
&\qquad = vv^{-1}(\gamma \otimes 1)v^{-1}v(1 \otimes \gamma)(\gamma \otimes \gamma) && \text{(since } \gamma \text{ is a symmetry for } v^{-1}) \\
&\qquad = 1 && \text{(as } \gamma \text{ is an involution)}
\end{aligned}$$

Conversely, if $v(\gamma\otimes\gamma)v(\gamma\otimes\gamma) = 1$ then $v(\gamma\otimes\gamma)v = (\gamma\otimes\gamma)$ since γ is an involution, and so $(\gamma\otimes\gamma)v(\gamma\otimes\gamma)v = 1$; that is: v has inverse $(\gamma \otimes \gamma)v(\gamma \otimes \gamma)$. ■

D.3 From a tricocycloid to an effect monoid

Lemma D.3 In a symmetric tricocycloid $H \in \mathbf{Set}$, for all $r, s, t, d, d' \in H$:

(vi) $(rs)^\perp (r \diamond s)^\perp = r^\perp$ **(viii)** $(r \diamond st)(s \diamond t)^\perp = (rs \diamond t)^\perp (r \diamond s)$ **(x)** $((s^\perp \diamond d')^\perp d)^\perp = ((sd)^\perp \diamond (s \diamond d)^\perp d')^\perp$

(vii) $(rs)^\perp \diamond (r \diamond s)^\perp = s^\perp$ **(ix)** $(r \diamond st) \diamond (s \diamond t)^\perp = ((rs \diamond t)^\perp \diamond (r \diamond s))^\perp$ **(xi)** $((s^\perp \diamond d')^\perp \diamond d)^\perp = (s \diamond d)^\perp \diamond d'$

Proof

- *Proof of (vi) and (vii)*: these assertions are a direct consequence of 2.1, but they can be proven in another way, by resorting to $(-)^\perp$ being involutive and v injective. We lay out this approach as well, as it will enable us to prove the other identities in a similar fashion. As $(-)^\perp$ is an involution and v an injection, it suffices to show that

$$\begin{cases} ((rs)^\perp (r \diamond s)^\perp)^\perp ((rs)^\perp \diamond (r \diamond s)^\perp) = rs^\perp \\ ((rs)^\perp (r \diamond s)^\perp)^\perp \diamond ((rs)^\perp \diamond (r \diamond s)^\perp) = r \diamond s^\perp \end{cases}$$

which is the case, because

$$((rs)^\perp (r \diamond s)^\perp)^\perp ((rs)^\perp \diamond (r \diamond s)^\perp) \overset{(iv)}{=} (rs)^\perp (r \diamond s) \overset{(iv)}{=} rs^\perp$$

$$((rs)^\perp (r \diamond s)^\perp)^\perp \diamond ((rs)^\perp \diamond (r \diamond s)^\perp) \overset{(v)}{=} ((rs)^\perp \diamond (r \diamond s))^\perp \overset{(v)}{=} r \diamond s^\perp$$

- *Proof of (viii) and (ix)*: similarly, it suffices to show that

$$\begin{cases} ((r \diamond st)(s \diamond t)^\perp)^\perp \cdot ((r \diamond st) \diamond (s \diamond t)^\perp) = ((rs \diamond t)^\perp (r \diamond s))^\perp \cdot ((rs \diamond t)^\perp \diamond (r \diamond s))^\perp \\ ((r \diamond st)(s \diamond t)^\perp)^\perp \diamond ((r \diamond st) \diamond (s \diamond t)^\perp) = ((rs \diamond t)^\perp (r \diamond s))^\perp \diamond ((rs \diamond t)^\perp \diamond (r \diamond s))^\perp \end{cases}$$

which is due to

$$\begin{aligned} ((r \diamond st)(s \diamond t)^\perp)^\perp \cdot ((r \diamond st) \diamond (s \diamond t)^\perp) &\overset{(iv)}{=} (r \diamond st)(s \diamond t) \overset{(ii)}{=} (rs \diamond t) \\ &\overset{(vi)}{=} ((rs \diamond t)^\perp (r \diamond s))^\perp \cdot ((rs \diamond t)^\perp \diamond (r \diamond s))^\perp \\ ((r \diamond st)(s \diamond t)^\perp)^\perp \diamond ((r \diamond st) \diamond (s \diamond t)^\perp) &\overset{(v)}{=} ((r \diamond st) \diamond (s \diamond t))^\perp \overset{(iii)}{=} (r \diamond s)^\perp \\ &\overset{(vii)}{=} ((rs \diamond t)^\perp (r \diamond s))^\perp \diamond ((rs \diamond t)^\perp \diamond (r \diamond s))^\perp \end{aligned}$$

- *Proof of (x) and (xi)*: first, we observe that

$$((\underbrace{s^\perp}_{\overset{(vi)}{=} (sd^\perp)^\perp (s \diamond d^\perp)^\perp} \diamond d')^\perp \underbrace{d}_{\overset{(vii)}{=} (sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp})^\perp \overset{(viii)}{=} (((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d')((s \diamond d^\perp)^\perp \diamond d')^\perp)^\perp$$

and

$$\begin{aligned} (s^\perp \diamond d')^\perp \diamond d)^\perp &= (((sd^\perp)^\perp (s \diamond d^\perp)^\perp \diamond d')^\perp \diamond (((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp))^\perp \\ &\overset{(ix)}{=} ((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d') \diamond ((s \diamond d^\perp)^\perp \diamond d')^\perp \end{aligned}$$

which enable us to conclude analogously, since:

$$\begin{aligned} ((s^\perp \diamond d')^\perp d)^\perp ((s^\perp \diamond d')^\perp \diamond d)^\perp &= (((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d')((s \diamond d^\perp)^\perp \diamond d')^\perp)^\perp \cdot (((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d') \diamond ((s \diamond d^\perp)^\perp \diamond d')^\perp) \\ &\overset{(iv)}{=} ((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d') \cdot ((s \diamond d^\perp)^\perp \diamond d') \overset{(ii)}{=} \underbrace{(sd^\perp)^\perp (s \diamond d^\perp)^\perp}_{\overset{(vi)}{=} s^\perp} \diamond d' \\ &\overset{(ii)}{=} ((sd)^\perp \diamond (s \diamond d)^\perp d')((s \diamond d)^\perp \diamond d') \end{aligned}$$

and

$$\begin{aligned}((s^\perp \diamond d')^\perp d)^\perp \diamond ((s^\perp \diamond d')^\perp \diamond d)^\perp &= \big(((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d')((s \diamond d^\perp)^\perp \diamond d')^\perp\big)^\perp \diamond \big(((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d') \diamond ((s \diamond d^\perp)^\perp \diamond d')^\perp\big) \\ &\overset{(v)}{=} \big(((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp d') \diamond ((s \diamond d^\perp)^\perp \diamond d')\big)^\perp \overset{(iii)}{=} ((sd^\perp)^\perp \diamond (s \diamond d^\perp)^\perp)^\perp \overset{(vii)}{=} d^\perp \\ &\overset{(vii)}{=} (sd)^\perp \diamond (s \diamond d)^\perp \overset{(iii)}{=} ((sd)^\perp \diamond (s \diamond d)^\perp d') \diamond ((s \diamond d)^\perp \diamond d')\end{aligned}$$

■

Corollary D.4 In a symmetric tricocycloid, for all s, d:

$$(s^\perp d)^\perp ((s^\perp \diamond d)^\perp d)^\perp = (sd)^\perp ((s \diamond d)^\perp d)^\perp$$

Proof
Since $s^\perp d = (sd)^\perp (s \diamond d)^\perp d$ by (vi) and $((s^\perp \diamond d)^\perp d)^\perp = ((sd)^\perp \diamond (s \diamond d)^\perp d)$ by (x), it comes that

$$(s^\perp d)^\perp ((s^\perp \diamond d)^\perp d)^\perp = ((sd)^\perp (s \diamond d)^\perp d)^\perp ((sd)^\perp \diamond (s \diamond d)^\perp d) \overset{(iv)}{=} (sd)^\perp ((s \diamond d)^\perp d)^\perp$$

■

A symmetric tricocycloid will be said to be **left-cancellative** if $\forall r, s, s'.\ rs = rs' \implies s = s'$ and satisfying the **double cancellation property** if $\forall r, s, r', s'.\ rs = s'r'$ and $rs^\perp = (s')^\perp r' \implies r = r'$.

Proposition D.3 In a symmetric tricocycloid where the double cancellation property holds:

$$\forall d, s.\ \ d^\perp = (sd)^\perp ((s \diamond d)^\perp d)^\perp$$

Proof
By double cancellation, since

$$(sd)^\perp ((s \diamond d)^\perp d)^\perp \cdot ((s \diamond d)^\perp \diamond d) \overset{(iv)}{=} (sd)^\perp ((s \diamond d)^\perp d)^\perp \cdot ((s \diamond d)^\perp \diamond d) = (sd)^\perp (s \diamond d)^\perp d^\perp \overset{(vi)}{=} s^\perp d^\perp$$

and

$$\begin{aligned}(sd)^\perp ((s \diamond d)^\perp d)^\perp \cdot ((s \diamond d)^\perp \diamond d)^\perp &\overset{\text{Corollary D.4}}{=} (s^\perp d)^\perp ((s^\perp \diamond d)^\perp d)^\perp \cdot ((s \diamond d)^\perp \diamond d)^\perp \\ &\overset{(xi)}{=} (s^\perp d)^\perp ((s^\perp \diamond d)^\perp d)^\perp \cdot ((s^\perp \diamond d)^\perp \diamond d) \overset{(iv)}{=} (s^\perp d)^\perp (s^\perp \diamond d)^\perp d^\perp \overset{(vi)}{=} sd^\perp\end{aligned}$$

■

D.3.1 From Effect Monoids to Tricocycloids

Lemma D.5 Let H be a set equipped with a function $v \colon H \times H \to H \times H, (r, s) \longmapsto (r \cdot s, r \diamond s)$ and an involution $\gamma \colon H \to H, r \longmapsto r^\perp$. Suppose that $\cdot$ is left cancellative. Then H is a symmetric tricocycloid iff $\cdot$ is associative and the axioms *Symmetry 1* and *Symmetry 1 orthogonal* hold.

Proof
By Lemma 2.1, it suffices to prove that the axioms *3-cocycle 1 & 2*, *Symmetry 2*, and *Symmetry 2 orthogonal* hold. We first prove that the axiom *3-cocycle 1* holds. Using axiom *Symmetry 1* three times we have

$$(rst)^\perp (r \diamond st)(s \diamond t) = r(st)^\perp (s \diamond t) = rst^\perp = (rst)^\perp (rs \diamond t)$$

and so $rs \diamond t = (r \diamond st)(s \diamond t)$ by left cancellativity.

We now prove axiom *3-cocycle 2*. We calculate that

$$\begin{aligned}(rs)^\perp \big((r \diamond st) \diamond (s \diamond t)\big) &= (rst)^\perp (rs \diamond t)^\perp \big((r \diamond st) \diamond (s \diamond t)\big) \\ &= (rst)^\perp \big((r \diamond st)(s \diamond t)\big)^\perp \big((r \diamond st) \diamond (s \diamond t)\big) \\ &= (rst)^\perp (r \diamond st)(s \diamond t)^\perp \\ &= r(st)^\perp (s \diamond t)^\perp = rs^\perp = (rs)^\perp (r \diamond s)\end{aligned}$$

using, in succession: axiom *Symmetry 1 orthogonal*, *3-cocycle 1*, axiom *Symmetry 1* twice, axiom *Symmetry 1 orthogonal*, and axiom *Symmetry 1*. So by left cancellativity we have $(r \diamond st) \diamond (s \diamond t) = r \diamond s$ as desired.

We next prove axiom *Symmetry 2*. By axiom *Symmetry 1* and axiom *Symmetry 1 orthogonal* twice, we have that $(r \diamond s^\perp)^\perp = (rs)^\perp \diamond (r \diamond s)$ for axiom *Symmetry 2*, it suffices by the defining property of the right hand side to prove that

$$(rs^\perp)^\perp\big((rs)^\perp \diamond (r \diamond s)\big) = (rs^\perp)^\perp (r \diamond s^\perp)^\perp = r^\perp = (rs)^\perp (r \diamond s)^\perp$$

$$(rs^\perp)^\perp (r \diamond s^\perp)^\perp = (rs)^\perp (r \diamond s)^\perp \ .$$

But

$$\begin{aligned}(rs^\perp)^\perp (r \diamond s^\perp)^\perp &= r^\perp \\ &= (rs)^\perp (r \diamond s)^\perp \\ &= \big((rs)^\perp (r \diamond s)\big)^\perp \big((rs)^\perp \diamond (r \diamond s)\big) \\ &= (rs^\perp)^\perp \big((rs)^\perp \diamond (r \diamond s)\big)\end{aligned}$$

by axiom *Symmetry 1 orthogonal* twice, then axiom *Symmetry 1* twice. Left cancellativity yields $(r \diamond s^\perp)^\perp = \big((rs)^\perp \diamond (r \diamond s)\big)$ as desired.

Finally, we prove axiom *Symmetry 2 orthogonal*. By axiom *Symmetry 1 orthogonal* and axiom *Symmetry 1* twice, we have

$$\begin{aligned}r\big((rs)^\perp \diamond (r \diamond s)^\perp\big) &= \big((rs)^\perp (r \diamond s)^\perp\big)^\perp \big((rs)^\perp \diamond (r \diamond s)^\perp\big) \\ &= (rs)^\perp (r \diamond s) = rs^\perp \ ;\end{aligned}$$

whence by left cancellativity that $\big((rs)^\perp \diamond (r \diamond s)^\perp\big) = s^\perp$ as desired. ■

D.4 Isomorphism of categories

Let $\mathcal{EMonNorm}$ be the category of effect monoids with normalisation and maps of effect algebras preserving multiplication, $\mathcal{TricoCanc}$ be the category of left-cancellative tricocycloids in Set with double cancellation and functions commuting with γ and v.

Theorem D.6 $\mathcal{EMonNorm} \cong \mathcal{TricoCanc}$ ■

Proof

The constructions $M \mapsto \mathring{M}$ and $H \mapsto \overline{H}$ of Lemma **??** and Lemma **??** yield two functors:

$$\mathring{(-)} \ := \begin{cases} \mathcal{EMonNorm} & \longrightarrow \mathcal{TricoCanc} \\ M \xrightarrow{f} M' & \longmapsto \mathring{M} \xrightarrow{f|_{\mathring{M}}} \mathring{M}' \end{cases} \quad \text{and} \quad \overline{(-)} \ := \begin{cases} \mathcal{TricoCanc} & \longrightarrow \mathcal{EMonNorm} \\ H \xrightarrow{\phi} H' & \longmapsto \overline{H} \xrightarrow{\bar{\phi}} \overline{H'} \end{cases}$$

where $\bar{\phi}|_H \ := \ \phi$ and $\bar{\phi}(0) \ := \ 0, \ \bar{\phi}(1) \ := \ 1$. Indeed:

- $f|_{\mathring{M}} \in \mathcal{TricoCanc}$ since it clearly preserves $\cdot$ and $\gamma = (-)^\perp$, as $1 = f(1) = f(r \ovee r^\perp) = f(r) \ovee f(r^\perp) \implies f(r^\perp) = f(r)^\perp$. Moreover, if $(r, s) \neq (1, 1)$ (whence $(f(r)f(s))^\perp \neq 0$), due to $(f(r)f(s))^\perp f(r \diamond s) = (f(rs))^\perp f(r \diamond s) = f((rs)^\perp) f(r \diamond s) = f((rs)^\perp (r \diamond s)) = f(rs^\perp) = f(r)f(s)^\perp$, it comes that $f(r \diamond s) = f(r) \diamond f(s)$ by Corollary D.1.
- $\bar{\phi} \in \mathcal{EMonNorm}$ as it preserves $\cdot$, 0 and 1. On top of that, if $rs = a \perp b = rs^\perp$, $\bar{\phi}(a \ovee b) = \bar{\phi}(r) = \bar{\phi}(r)\bar{\phi}(s) \ovee \bar{\phi}(r)\bar{\phi}(s)^\perp = \bar{\phi}(r)\bar{\phi}(s) \ovee \bar{\phi}(r)\bar{\phi}(s^\perp) = \bar{\phi}(a) \ovee \bar{\phi}(b)$ by right distributivity, and due to ϕ preserving multiplication and $(-)^\perp$.

- $(\mathring{-})$ and $\overline{(-)}$ clearly preserve identity morphisms and composition.

Let us show that, for all $M \in$ ***EMonNorm*** and $H \in$ ***TricoCanc***

- $(M, \cdot, \varovee, (-)^{\perp}) = (\overline{\mathring{M}}, \cdot', \varovee', (-)^{\perp'})$: Clearly $\cdot = \cdot'$ and $(-)^{\perp} = (-)^{\perp'}$. Furthermore, let $\overline{\mathring{M}} = M \ni a \perp b$. If $a \varovee b = 0$, then $a \varovee b \perp 1$, thus $a \perp 1$ and $b \perp 1$, whence $a = b = 0$.
  Else, $\begin{cases} 1 \neq (a \varovee b)^{\perp} \perp a & \text{so } \exists c;\ a = (a \varovee b)c \\ 1 \neq (a \varovee b)^{\perp} \perp b & \text{so } \exists c';\ a = (a \varovee b)c' \end{cases}$ by normalisation. Due to right distributivity, we then have $0 \neq a \varovee b = (a \varovee b)(c \varovee c')$, and by Corollary D.1, $c \varovee c' = 1$, *i.e.* $c' = c^{\perp}$. As a result, $a \varovee' b \ := \ a \varovee b$.
- $(H, \cdot, \diamond, (-)^{\perp}) = (\mathring{\overline{H}}, \cdot', \diamond', (-)^{\perp'})$: Clearly $\cdot = \cdot'$ and $(-)^{\perp} = (-)^{\perp'}$. For every $r, s \in H = \mathring{\overline{H}}$, let us show that $r \diamond' s = r \diamond s$. But by definition of $r \diamond' s$: $(rs)^{\perp}(r \diamond' s) = rs^{\perp} = (rs)^{\perp}(r \diamond s)$. We conclude by left-cancellativity.

$\blacksquare$

E. Effectus-theoretic lemmas

E.1 Pullback lemmas

Reminder

To begin with, we recall well-known results about pullbacks:

Proposition E.1 — Pasting law for pullbacks In any category, a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, "i"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "j"] \\ D \arrow[r, "k"'] & E \arrow[r, "l"'] & F \end{tikzcd}$$

where the right-hand inner square is a pullback is such that: the left-hand square is a pullback iff the outer one is.

Proof

- $\Longrightarrow$: let $D \xleftarrow{\phi} X \xrightarrow{\psi} C$ be a cone of $D \xrightarrow{lk} F \xleftarrow{j} C$. The right pullback induces a unique cone map $X \xrightarrow{u} B$, which in turn, as the left square is a pullback, induces a unique cone map $X \xrightarrow{v} A$. Thus, $\phi = hv$ and $gfv = gu = \psi$, and any other cone morphism $v' \colon X \to A$ equals v by unicity since $hv' = \phi$ and $fv' = u$, as $i(fv') = khv' = k\phi$ and $g(fv') = \psi$.
- $\Longleftarrow$: let $D \xleftarrow{\phi} X \xrightarrow{\psi} B$ be a cone of $D \xrightarrow{k} E \xleftarrow{i} B$. The outer pullback induces a unique map $X \xrightarrow{u} A$, and the right pullback ensures that ψ is the terminal cone map from $E \xleftarrow{k\phi} X \xrightarrow{g\psi} C$ to $E \xleftarrow{i} B \xrightarrow{g} C$. Thus u is a cone map for the left square, since $hu = \phi$ and $fu = \psi$, as $i(fu) = khu = k\phi = \psi$ and $g(fu) = (gf)u = g\psi$ (by unicity of ψ). And it is unique since any other cone map for the left square $X \xrightarrow{u'} A$ satisfies $hu' = \phi$ and $(gf)u' = g(fu') = g\psi$.

■

Proposition E.2 — Isomorphism square A diagram of the following form is a pullback

$$\begin{tikzcd} A' \arrow[r, "g", "\cong"'] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r, "g'", "\cong"'] & B \end{tikzcd}$$

Proof

Let $B' \xleftarrow{\phi} X \xrightarrow{\psi} A$ be a cone of $B' \xrightarrow{g'} B \xleftarrow{f} A$. Any morphism of cone $u \colon X \to A'$ is uniquely determined by $u = g^{-1}\psi$. On top of that, $g^{-1}h \colon X \to A'$ is indeed a morphism of cone, since $g(g^{-1}\psi) = \psi$ and $f'(g^{-1}\psi) = g'^{-1} f\psi = \phi$. ■

On top of that, recall these well-known properties of the coproduct (the proofs are straightforward):

$$[gf_1, gf_2] = g[f_1, f_2] \qquad [g_1 f_1, g_2 f_2] = [g_1, g_2](f_1 + f_2) \qquad \begin{gathered}[f_1, f_2]\kappa_i = f_i \\ (f_1 + f_2)\kappa_i = \kappa_i f_i\end{gathered} \qquad (g_1 + g_2)(f_1 + f_2) = g_1 f_1 + g_2 f_2$$

Now, we give a standalone exposition of a handful of effectus-theoretic lemmas by Bart Jacobs and Kenta Cho (see [Jac15]) that we will make use of (with detailed proofs, which are often omitted in the literature). In what follows, we fix an effectus $\mathbb{B}$.

Lemma E.1 In $\mathbb{B}$, squares of the following form are pullbacks:

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\kappa_1"'] & X \arrow[d, "\kappa_1"] \\
X + A \arrow[r, "g+f"'] & X + B
\end{tikzcd}
$$

Proof
First, note that the result holds for $f = \mathrm{id}$:

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\kappa_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\kappa_1"] \\
X + A \arrow[r, "g+1"'] & X + A
\end{tikzcd}
$$

owing to Proposition E.1, Proposition E.2 and eq. (3.1a), as we have pullbacks:

$$
\begin{tikzcd}
A \arrow[r, "\kappa_2", "\cong"'] \arrow[d, "\kappa_1"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rrr, bend left, "g"] & 0 + A \arrow[r, "1+g"] \arrow[d, "!+1"] \arrow[dr, phantom, "\lrcorner", very near start] & 0 + B \arrow[r, "{[!,1]}", "\cong"'] \arrow[d, "!+1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\kappa_1"] \\
A + X \arrow[r, "\times", "\cong"'] \arrow[rrr, bend right, "g+1"'] & X + A \arrow[r, "1+g"'] & X + B \arrow[r, "\times", "\cong"'] & B + X
\end{tikzcd}
$$

Now, modulo Proposition E.1, we get the desired pullback by pasting the previous $f = \mathrm{id}$ one and the eq. (3.1b) one:

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "\kappa_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, equal] \arrow[d, "\kappa_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\kappa_1"] \\
A + X \arrow[r, "g+1"'] & B + X \arrow[r, "1+f"'] & B + Y
\end{tikzcd}
$$

■

Lemma E.2 In $\mathbb{B}$, squares of the form:

$$
\begin{tikzcd}
(A + X) + 1 \arrow[r, "(g+1)+1"] \arrow[d, "{[\rhd_1,\kappa_2]}"'] & (B + X) + 1 \arrow[d, "{[\rhd_1,\kappa_2]}"] \\
A + 1 \arrow[r, "g+1"'] & B + 1
\end{tikzcd}
\qquad
\begin{tikzcd}
(X + A) + 1 \arrow[r, "(1+g)+1"] \arrow[d, "{[\rhd_2,\kappa_2]}"'] & (X + B) + 1 \arrow[d, "{[\rhd_2,\kappa_2]}"] \\
A + 1 \arrow[r, "g+1"'] & B + 1
\end{tikzcd}
\tag{E.1a, E.1b}
$$

are pullbacks.

Proof
`Proof of E.1a`: Let $A+1 \xleftarrow{\phi} Z \xrightarrow{\psi} (B+X)+1$ be a cone of the cospan $A+1 \xrightarrow{g+1} B+1 \xleftarrow{[\rhd_1,\kappa_2]} (B+X)+1$. The following 3.1a-shaped pullback induces a unique cone morphism $k \colon Z \to A + (X+1)$:

$$
\begin{tikzcd}
Z \arrow[drr, dotted, "k"] \arrow[drrrr, bend left=10, "\alpha \circ \psi"] \arrow[ddrr, bend right, "\phi"'] & & & & \\
& & A + (X + 1) \arrow[rr, "g+1"] \arrow[d, "1+!"'] \arrow[dr, phantom, "\lrcorner", very near start] & & B + (X + 1) \arrow[d, "{[\rhd_1, \kappa_2] \circ \alpha^{-1} = 1+!}"] \\
& & A + 1 \arrow[rr, "g+1"'] & & B + 1
\end{tikzcd}
$$

It comes that $k' := \alpha^{-1} \circ k \colon Z \to (A + X) + 1$

$$
\begin{tikzcd}
Z \arrow[drr, dotted, "k' := \alpha^{-1} \circ k"] \arrow[drrr, bend left=10, "\psi"] \arrow[ddrr, bend right, "\phi"'] & & & \\
& & (A + X) + 1 \arrow[r, "(g+1)+1"] \arrow[d, "{[\rhd_1, \kappa_2]}"] & (B + X) + 1 \arrow[d, "{[\rhd_1, \kappa_2]}"] \\
& & A + 1 \arrow[r, "g+1"'] & B + 1
\end{tikzcd}
$$

is a cone morphism for the original cospan, as

$$((g + 1) + 1)k' = ((g + 1) + 1)\alpha^{-1} k = \alpha^{-1} (g + 1)k = \alpha^{-1} \alpha \psi = \psi$$

and

$$[\rhd_1, \kappa_2]k' = [\rhd_1, \kappa_2]\alpha^{-1} k = (1+!)k = \phi$$

On top of that, it is unique, since any other cone morphism $k'' \colon Z \to (A + X) + 1$ is such that $(1+!)\big(\alpha k''\big) = [\rhd_1, \kappa_2]k'' = \phi$ and $(g + 1)\big(\alpha k''\big) = \alpha((g + 1) + 1)k'' = \alpha\psi$, so that $\alpha k'' = k$ by unicity, and $k'' = \alpha^{-1} k = k''$.

`Proof of E.1b`: this results from Proposition E.2, Proposition E.1, and diagrams of the form E.1a being pullbacks, since any diagram of the form E.1b can be written:

$$
\begin{tikzcd}
(X + A) + 1 \arrow[r, "\times_{+1}", "\cong"'] \arrow[d, "{[\rhd_2, \kappa_2]}"'] \arrow[rrr, bend left, "(1+g)+1"] \arrow[dr, phantom, "\lrcorner", very near start] & (A + X) + 1 \arrow[r, "(g+1)+1"] \arrow[d, "{[\rhd_1, \kappa_2]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (B + X) + 1 \arrow[r, "\times_{+1}", "\cong"'] \arrow[d, "{[\rhd_1, \kappa_2]}"] \arrow[dr, phantom, "\lrcorner", very near start] & (X + B) + 1 \arrow[d, "{[\rhd_2, \kappa_2]}"] \\
A + 1 \arrow[r, "1", "\cong"'] \arrow[rrr, bend right, "g+1"'] & A + 1 \arrow[r, "g+1"'] & B + 1 \arrow[r, "1", "\cong"'] & B + 1
\end{tikzcd}
$$

■

Lemma E.3 For every $X \in \mathbb{B}$ and $n \geq 2$, the maps $[\triangleright_i, \kappa_2] \colon X_1 + \cdots + X_n + 1 \to X_i + 1$, for $1 \leq i \leq n$, are jointly monic.

Proof

NB To avoid confusion, in this proof, we will denote $\triangleright_i \colon X_1 + \cdots + X_n \to X_i + 1$ more explicitly by $\triangleright_i^{X_1, \ldots, X_n}$.

We use an analogous argument to the one due to Kenta Cho in [Jac15, Lemma 4.2.].

We proceed by induction.

`Base case`: Let's first show the property for $n = 2$, *i.e.* the maps $[\triangleright_1^{X,Y}, \kappa_2] \colon (X+Y)+1 \to X+1$, $[\triangleright_2^{X,Y}, \kappa_2] \colon (X+Y)+1 \to Y+1$ are jointly monic. Let $f, g \colon Z \to (X+Y)+1$ such that $[\triangleright_i^{X,Y}, \kappa_2]\, f = [\triangleright_i^{X,Y}, \kappa_2]\, g$, for $i = 1, 2$.

Consider the following diagram:

$$\begin{tikzcd}
Z \arrow[dr, bend left, "f"] \arrow[dr, bend right, "g"'] & & & \\
& (X+Y)+1 \arrow[r, "(!+1)+1"] \arrow[d, "(1+!)+1"] \arrow[dd, bend right=60, "{[\triangleright_1^{X,Y}, \kappa_2]}"'] \arrow[rr, bend left, "{[\triangleright_2^{X,Y}, \kappa_2]}"] \arrow[dr, phantom, "\lrcorner", very near start] & (1+Y)+1 \arrow[r, "{[\triangleright_2^{1,Y}, \kappa_2]}"] \arrow[d, "(1+!)+1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y+1 \arrow[d, "!+1"] \\
& (X+1)+1 \arrow[r, "(!+1)+1"'] \arrow[d, "{[\triangleright_1^{X,1}, \kappa_2]}"] \arrow[dr, phantom, "\lrcorner", very near start] & (1+1)+1 \arrow[r, "\backslash\!\!\backslash\!/"'] \arrow[d, "\|\!/"] & 1+1 \\
& X+1 \arrow[r, "!+1"'] & 1+1 &
\end{tikzcd} \tag{E.2}$$

In Lemma E.2, E.1a (resp. E.1b) causes the lower left (resp. the upper right) square to a pullback. The upper left is a pullback owing to Proposition E.2, eq. (3.1a), Proposition E.1, and the fact that it can be written:

$$\begin{tikzcd}
(X+Y)+1 \arrow[r, "\cong"'] \arrow[d, "(1+!)+1"'] \arrow[rrr, bend left, "(!+1)+1"] \arrow[dr, phantom, "\lrcorner", very near start] & X+(Y+1) \arrow[r, "!+1"] \arrow[d, "1+(!+1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1+(Y+1) \arrow[r, "\cong"'] \arrow[d, "1+(!+1)"] \arrow[dr, phantom, "\lrcorner", very near start] & (1+Y)+1 \arrow[d, "(1+!)+1"] \\
(X+1)+1 \arrow[r, "\cong"'] \arrow[rrr, bend right, "(!+1)+1"'] & X+(1+1) \arrow[r, "!+1"'] & 1+(1+1) \arrow[r, "\cong"'] & (1+1)+1
\end{tikzcd}$$

From $[\triangleright_i^{X,Y}, \kappa_2]\, f = [\triangleright_i^{X,Y}, \kappa_2]\, g$, we get

$$\underbrace{(!+1)[\triangleright_i^{X,Y}, \kappa_2]}\, f = (!+1)[\triangleright_i^{X,Y}, \kappa_2]\, g$$

$$\overset{\text{by } E.2}{=} \begin{cases} \|\!/ \circ ((!+!)+1) & \text{if } i = 1 \\ \backslash\!\!\backslash\!/ \circ ((!+!)+1) & \text{if } i = 2 \end{cases}$$

Therefore, by joint monicity of $\|\!/$ and $\backslash\!\!\backslash\!/$ in the effectus $\mathbb{B}$: $((!+!)+1)\, f = ((!+!)+1)\, g$. It comes that

- $((!+1)+1)((1+!)+1)\,f = ((!+1)+1)((1+!)+1)\,g$ results in $((1+!)+1)\,f$ and $((1+!)+1)\,g$ being cone morphisms from

$$X+1 \xleftarrow{[\rhd_1^{X,Y},\kappa_2]\,f \,=\, [\rhd_1^{X,Y},\kappa_2]\,g} Z \xrightarrow{((!+!)+1)\,f \,=\, ((!+!)+1)\,g} (1+1)+1$$

  to

$$X+1 \xleftarrow{[\rhd_1^{X,1},\kappa_2]} (X+1)+1 \xrightarrow{(!+1)+1} (1+1)+1$$

  so, by the lower left pullback in E.2, they are equal: $((1+!)+1)\,f = ((1+!)+1)\,g$
- similarly, $((1+!)+1)((!+1)+1)\,f = ((1+!)+1)((!+1)+1)\,g$ results in $((!+1)+1)\,f$ and $((!+1)+1)\,g$ being cone morphisms from

$$(1+1)+1 \xleftarrow{((!+!)+1)\,f \,=\, ((!+!)+1)\,g} Z \xrightarrow{[\rhd_2^{X,Y},\kappa_2]\,f \,=\, [\rhd_2^{X,Y},\kappa_2]\,g} Y+1$$

  to

$$(1+1)+1 \xleftarrow{(1+!)+1} (1+Y)+1 \xrightarrow{[\rhd_2^{1,Y},\kappa_2]} Y+1$$

  so, by the upper right pullback in E.2, they are equal: $((!+1)+1)\,f = ((!+1)+1)\,g$

Consequently, f and g are cone morphisms from

$$(X+1)+1 \xleftarrow{((1+!)+1)\,f=((1+!)+1)\,g} Z \xrightarrow{((!+1)+1)\,f=((!+1)+1)\,g} (1+Y)+1$$

to

$$(X+1)+1 \xleftarrow{(1+!)+1} (X+Y)+1 \xrightarrow{(!+1)+1} (1+Y)+1$$

therefore, by the upper left pullback in E.2, they are equal: $f = g$.

`Inductive step`: Suppose the result for all $k \leq n$; we will prove it for $n+1$, *i.e.* the maps $[\rhd_i^{X_1,\dots,X_{n+1}}, \kappa_2] \colon X_1 + \cdots + X_{n+1} + 1 \to X_i + 1$, for $1 \leq i \leq n+1$, are jointly monic.

Let $f, g \colon Z \to X_1 + \cdots + X_{n+1} + 1$ such that $[\rhd_i^{X_1,\dots,X_{n+1}}, \kappa_2]\, f = [\rhd_i^{X_1,\dots,X_{n+1}}, \kappa_2]\, g$ for $1 \leq i \leq n+1$. Then:

$$\forall i \leq n,\ \underbrace{[\rhd_i^{X_1,\dots,X_{n+1}}, \kappa_2]}_{[\rhd_i^{X_1,\dots,X_n},\kappa_2][\rhd_1^{X_1+\cdots+X_n,X_{n+1}},\kappa_2]}\ f = [\rhd_i^{X_1,\dots,X_{n+1}}, \kappa_2]\, g$$

And by joint monicity of $[\rhd_i^{X_1,\dots,X_n}, \kappa_2]$ (for $1 \leq i \leq n$), it comes that

$$[\rhd_1^{X_1+\cdots+X_n,X_{n+1}}, \kappa_2]\, f = [\rhd_1^{X_1+\cdots+X_n,X_{n+1}}, \kappa_2]\, g$$

We conclude by joint monicity of $[\rhd_1^{X_1+\cdots+X_n,X_{n+1}}, \kappa_2]$ and $[\rhd_2^{X_1+\cdots+X_n,X_{n+1}}, \kappa_2] = [\rhd_{n+1}^{X_1,\dots,X_{n+1}}, \kappa_2]$. ■

Lemma E.4 In $\mathbb{B}$, the coprojections $\kappa_1 \colon X \to X + A,\ \kappa_2 \colon X \to A + X$ are monic.

Proof

κ_1 is monic due to Proposition E.2, eq. (3.1b), Proposition E.1, and the following pullback diagram:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r, equal] \arrow[d, "\kappa_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\kappa_1"] \\
A \arrow[r, "\kappa_1", "\cong"'] \arrow[rr, bend right, "\kappa_1"'] & A+0 \arrow[r, "1+!"] & A+X
\end{tikzcd}$$

From the previous outer pullback diagram, Proposition E.2, and Proposition E.1, we get that κ_2 is monic too:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r, equal] \arrow[d, "\kappa_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\kappa_2"] \\
A \arrow[r, "\kappa_1"] \arrow[rr, bend right, "\kappa_2"'] & A + X \arrow[r, "\times", "\cong"'] & X + A
\end{tikzcd}$$

■

E.2 Effect algebra of predicates and Effect monoid of scalars

Theorem E.5 In $\mathbb{B}$, the predicates $\mathrm{Pred}(X) = \mathrm{Hom}_{\mathbb{B}}(X, 2)$ form an effect algebra. ■

Proof
For all predicates $p, q \colon X \to 2$, we define

- $p \perp q \quad \overset{\text{def}}{\Longleftrightarrow} \quad$ there exists $b \colon 1 \to 3$ called a **bound** such that $\begin{cases} |\!/\; b = p \\ \times\!/\; b = q \end{cases}$
- if $|\!/\; b = p \perp q = \times\!/\; b, \qquad p \ovee q \;:=\; (\nabla + \mathrm{id})\, b$
- $0 \;:=\; \kappa_2\,!$ and $1 \;:=\; \kappa_1\,!$
- $p^{\perp} \;:=\; \times\, p$

With these definitions, let us show that we have an effect algebra:

- `PCM structure`:
  - $p \ovee q = q \ovee p$ via the bound $\times|\; b$, since $|\!/\,\times| = \times\!/$ and $\times\!/\,\times| = |\!/$
  - $0 \ovee p = p$ via the bound $(\kappa_2 + 1)p$, since $|\!/\,(\kappa_2+1)p = [[\kappa_1, \kappa_2], \kappa_2](\kappa_2+1)p = [\kappa_2, \kappa_2]p = \kappa_2 \nabla p = \kappa_2! = 0$ and $\times\!/\,(\kappa_2 + 1)p = [[\kappa_2, \kappa_1], \kappa_2](\kappa_2 + 1)p = [\kappa_1, \kappa_2]p = p$.
  - if $\begin{cases} |\!/\, a = p \perp q = \times\!/\, a \\ |\!/\, b = (p \ovee q) \perp r = \times\!/\, b \end{cases}$, let's show that $\begin{cases} q \perp r \\ p \perp (q \ovee r) \end{cases}$ and $(p \ovee q) \ovee r = p \ovee (q \ovee r)$. Indeed, consider the following 3.1a-shaped pullback:

$$\begin{tikzcd}
X \arrow[dr, dotted, "c"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "|||_b"'] & & \\
& (1+1)+(1+1) \arrow[r, "1+\nabla"] \arrow[d, "\nabla+1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (1+1)+1 \arrow[d, "\nabla+1"] \\
& 1+(1+1) \arrow[r, "1+\nabla"'] & 1+1
\end{tikzcd}$$

By putting

$$\begin{aligned}
|||\; &:= [\kappa_1, \kappa_2 + 1] \\
|||\; &:= [[\kappa_2, \kappa_1 \circ \kappa_1], \kappa_2 + 1]
\end{aligned}$$

it comes that

$$c' := X \xrightarrow{c} (1+1)+(1+1) \xrightarrow{\mathbb{W}} (1+1)+1 \text{ is a bound for } q \perp r$$

since

$$\begin{aligned}
{\color{red}[[\kappa_1,\kappa_2],\kappa_2]}[[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c &= [[[{\color{red}[[\kappa_1,\kappa_2],\kappa_2]}\kappa_2, {\color{red}[[\kappa_1,\kappa_2],\kappa_2]}\kappa_1\circ\kappa_1],[[\kappa_1,\kappa_2]\kappa_2,\kappa_2]]c \\
&= [[\kappa_2,[\kappa_1,\kappa_2]\circ\kappa_1],[\kappa_2,\kappa_2]]c = [[\kappa_2,\kappa_1],\kappa_2\underbrace{[1,1]}_{\nabla}]c \\
&= [[\kappa_2,\kappa_1],\kappa_2](1+\nabla)c = [[\kappa_2,\kappa_1],\kappa_2]b = q \\
{\color{red}[[\kappa_2,\kappa_1],\kappa_2]}[[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c &= [[[{\color{red}[[\kappa_2,\kappa_1],\kappa_2]}\kappa_2, {\color{red}[[\kappa_2,\kappa_1],\kappa_2]}\kappa_1\circ\kappa_1],[[\kappa_2,\kappa_1]\kappa_2,\kappa_2]]c \\
&= [[\kappa_2,\kappa_2],[\kappa_1,\kappa_2]]c = [\kappa_2\nabla,1]c \\
&= [\kappa_2,1](\nabla+1)c = [\kappa_2,1]\Updownarrow\Updownarrow\Updownarrow b = [\kappa_2,1][1+\kappa_1,\kappa_2\kappa_2]b \\
&= [[\kappa_2,1](1+\kappa_1),[\kappa_2,1]\kappa_2\kappa_2]b \\
&= [[\kappa_2,\kappa_1],\kappa_2]b = r
\end{aligned}$$

$$c'' := X \xrightarrow{c} (1+1)+(1+1) \xrightarrow{\mathbb{W}} (1+1)+1 \text{ is a bound for } p \perp (q \ovee r)$$

since

$$\begin{aligned}
{\color{red}[[\kappa_1,\kappa_2],\kappa_2]}[\kappa_1,\kappa_2+1]c &= [{\color{red}[[\kappa_1,\kappa_2],\kappa_2]}\kappa_1,[[\kappa_1,\kappa_2]\kappa_2,\kappa_2]]c \\
&= [[\kappa_1,\kappa_2],[\kappa_2,\kappa_2]]c = [[\kappa_1,\kappa_2],\kappa_2\nabla]c \\
&= [[\kappa_1,\kappa_2],\kappa_2](1+\nabla)c = [[\kappa_1,\kappa_2],\kappa_2]a = p \\
{\color{red}[[\kappa_2,\kappa_1],\kappa_2]}[\kappa_1,\kappa_2+1]c &= [{\color{red}[[\kappa_2,\kappa_1],\kappa_2]}\kappa_1,[[\kappa_2,\kappa_1]\kappa_2,\kappa_2]]c \\
&= [[\kappa_2,\kappa_1],[\kappa_1,\kappa_2]]c \\
&= [[\kappa_2,[[\kappa_1,\kappa_1],\kappa_2]\kappa_1\circ\kappa_1],[[\kappa_1,\kappa_1]\kappa_2,\kappa_2]]c \\
&= [[[[\kappa_1,\kappa_1],\kappa_2]\kappa_2,[[\kappa_1,\kappa_1],\kappa_2]\kappa_1\circ\kappa_1],[[\kappa_1,\kappa_1]\kappa_2,\kappa_2]]c \\
&= [[[\kappa_1,\kappa_1],\kappa_2][\kappa_2,\kappa_1\circ\kappa_1],[[\kappa_1,\kappa_1],\kappa_2](\kappa_2+1)]c \\
&= [[\kappa_1,\kappa_1],\kappa_2][[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c \\
&= [\kappa_1[\mathrm{id},\mathrm{id}],\kappa_2][[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c \\
&= [[\kappa_2,\kappa_1]\kappa_2[\mathrm{id},\mathrm{id}],\kappa_2][[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c \\
&= [[\kappa_2,\kappa_1][\kappa_2,\kappa_2],\kappa_2][[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c \\
&= [[\kappa_2,\kappa_1]\kappa_2[\mathrm{id},\mathrm{id}],\kappa_2][[\kappa_2,\kappa_1\circ\kappa_1],\kappa_2+1]c \\
&= [[\kappa_2,\kappa_1]\kappa_2[\mathrm{id},\mathrm{id}],\kappa_2]c' \\
&= [[\kappa_2,\kappa_1],\kappa_2](\nabla+1)c' = q \ovee r
\end{aligned}$$

And

$$\begin{aligned}
(p \ovee q) \ovee r = (\nabla+1)b &= (\nabla+1)\overbrace{\Updownarrow\Updownarrow\Updownarrow}^{[\kappa_1\circ\kappa_1,\,\kappa_2+\mathrm{id}]}(\nabla+1)c \\
&= (\nabla+1)\,[\kappa_1\circ\kappa_1\circ\nabla,\ \kappa_2+\mathrm{id}]c = [(\nabla+1)\kappa_1\circ\kappa_1\circ\nabla,\ (\nabla+1)(\kappa_2+\mathrm{id})]c \\
&= [\kappa_1\circ\nabla\circ\kappa_1\circ\nabla,\ (\nabla+1)(\kappa_2+\mathrm{id})]c = [\kappa_1\circ 1\circ\nabla,\ (\nabla+1)(\kappa_2+\mathrm{id})]c \\
&= [\kappa_1\circ 1\circ\nabla,\ (\nabla+1)(\kappa_2+\mathrm{id})]c = [(\nabla+1)\kappa_1,\ (\nabla+1)(\kappa_2+\mathrm{id})]c \\
&= (\nabla+1)\mathbb{W}c = p \ovee (q \ovee r)
\end{aligned}$$

- `Effect algebra structure:`
  - $\kappa_1\, p$ is a bound for $p \perp p^{\perp}$

- If $\rhd_1 b = p \perp q = \rhd_2 b$, let us show that $p \ovee q = 1 \ := \ \kappa_1!$ implies that $q = p^\perp \ := \ \times p$. Indeed, consider the following pullback (by Lemma E.1):

$$\begin{tikzcd}
X \arrow[dr, dotted, "c"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "!"'] & & \\
& 1+1 \arrow[r, "\kappa_1"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\lrcorner", very near start] & (1+1)+1 \arrow[d, "\nabla+1"] \\
& 1 \arrow[r, "\kappa_1"'] & 1+1
\end{tikzcd}$$

Then

$$q = \rhd_2 b = \rhd_2 \kappa_1 c = \times \rhd_1 \kappa_1 c = \times \rhd_1 b = p^\perp$$

- if $\rhd_1 b = \kappa_1! = 1 \perp p = \rhd_2 b$, let us show that $p = 0 \ := \ \kappa_2!$. Indeed, by $3.1b$, we have

$$\begin{tikzcd}
X \arrow[r, bend left, "b"] \arrow[dr, dotted, "!"] \arrow[ddr, bend right, "!"'] & (1+1)+1 \arrow[dr, bend left, "\alpha"] & \\
& 1 \arrow[r, "\kappa_1"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & 1+(1+1) \arrow[d, "1+\nabla"] \\
& 1 \arrow[r, "\kappa_1"'] & 1+1
\end{tikzcd}$$

Then

$$b = \overbrace{\alpha^{-1}}^{[\kappa_1 \circ \kappa_1, \kappa_2 + \mathrm{id}]} \kappa_1! = \kappa_1 \circ \kappa_1!$$

and

$$p = \rhd_2 b = [[\kappa_2, \kappa_1], \kappa_2] \, \kappa_1 \circ \kappa_1! = \kappa_1!$$

■

Theorem E.6 In $\mathbb{B}$, the scalars $M_{\mathbb{B}} \ := \ \mathrm{Pred}(1) = \mathrm{Stat}(2) = \mathrm{Hom}_{\mathbb{B}}(1, 2)$ form an effect monoid. ■

Proof
By Theorem E.5, they form an effect algebra. We define, for all $r, s\colon 1 \to 2$:

$$r \cdot s \ := \ 1 \xrightarrow{r} 2 \xrightarrow{[s, \kappa_2]} 2$$

NB Note that, for a technical reason (this will leat to a neater formulation in terms of tricocycloids), we take the definition set in [Jac11, Proposition 3.1.], which is dual to the one of [Jac15], but the result holds similarly in both cases.

Let $s, r, r'\colon 1 \to 2$ be scalars such that $\rhd_1 b = r \perp r' = \rhd_2 b$.
Then

$$c \;:=\; 1 \xrightarrow{s} 1+1 \xrightarrow{b+\mathrm{id}} 3+1 \xrightarrow{[\mathrm{id},\kappa_2]} 3 \text{ is a bound for } s\cdot r \perp s\cdot r'$$

since

$$\begin{aligned}
[[\kappa_1,\kappa_2],\kappa_2]c &= [[\kappa_1,\kappa_2],\kappa_2][\mathrm{id},\kappa_2](b+\mathrm{id})s \\
&= [[[\kappa_1,\kappa_2],\kappa_2],[[\kappa_1,\kappa_2],\kappa_2]\kappa_2](b+\mathrm{id})s \\
&= [[[\kappa_1,\kappa_2],\kappa_2],\kappa_2](b+\mathrm{id})s \\
&= [[[\kappa_1,\kappa_2],\kappa_2]b,\kappa_2]s = [r,\kappa_2]s \\
[[\kappa_2,\kappa_1],\kappa_2]c &= [[\kappa_1,\kappa_2],\kappa_2][\mathrm{id},\kappa_2](b+\mathrm{id})s \\
&= [[[\kappa_2,\kappa_1],\kappa_2],[[\kappa_2,\kappa_1],\kappa_2]\kappa_2](b+\mathrm{id})s \\
&= [[[\kappa_2,\kappa_1],\kappa_2],\kappa_2](b+\mathrm{id})s \\
&= [[[\kappa_2,\kappa_1],\kappa_2]b,\kappa_2]s = [r',\kappa_2]s
\end{aligned}$$

$$d \;:=\; 1 \xrightarrow{b} 1+1 \xrightarrow{(s+s)+\mathrm{id}} 3+1 \xrightarrow{[[\kappa_1+\mathrm{id},\kappa_2+\mathrm{id}],\kappa_2]} 3 \text{ is a bound for } r\cdot s \perp r'\cdot s$$

since

$$\begin{aligned}
[[\kappa_1,\kappa_2],\kappa_2]d &= [[\kappa_1,\kappa_2],\kappa_2][[\kappa_1+\mathrm{id},\kappa_2+\mathrm{id}],\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[\kappa_1,\kappa_2],\kappa_2][\kappa_1+\mathrm{id},\kappa_2+\mathrm{id}],[[\kappa_1,\kappa_2],\kappa_2]\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[[\kappa_1,\kappa_2],\kappa_2](\kappa_1+\mathrm{id}),[[\kappa_1,\kappa_2],\kappa_2](\kappa_2+\mathrm{id})],\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[\kappa_1,\kappa_2],[\kappa_2,\kappa_2]],\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[\kappa_1,\kappa_2],[\kappa_2,\kappa_2]](s+s),\kappa_2]b \\
&= [[[\kappa_1,\kappa_2]s,[\kappa_2,\kappa_2]s],\kappa_2]b \\
&= [[s,\kappa_2\,\underbrace{[\mathrm{id},\mathrm{id}]s}_{=\mathrm{id}\,:\,1\to 1}],\kappa_2]b \\
&= [[s,\kappa_2],\kappa_2]b \\
&= [[[s,\kappa_2]\kappa_1,[s,\kappa_2]\kappa_2],\kappa_2]b \\
&= [[s,\kappa_2][\kappa_1,\kappa_2],[s,\kappa_2]\kappa_2]b \\
&= [s,\kappa_2][[\kappa_1,\kappa_2],\kappa_2]b = [s,\kappa_2]r \\
[[\kappa_2,\kappa_1],\kappa_2]d &= [[\kappa_2,\kappa_1],\kappa_2][[\kappa_1+\mathrm{id},\kappa_2+\mathrm{id}],\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[\kappa_2,\kappa_1],\kappa_2][\kappa_1+\mathrm{id},\kappa_2+\mathrm{id}],[[\kappa_2,\kappa_1],\kappa_2]\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[[\kappa_2,\kappa_1],\kappa_2](\kappa_1+\mathrm{id}),[[\kappa_2,\kappa_1],\kappa_2](\kappa_2+\mathrm{id})],\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[\kappa_2,\kappa_2],[\kappa_1,\kappa_2]],\kappa_2]((s+s)+\mathrm{id})b \\
&= [[[\kappa_2,\kappa_2]s,[\kappa_1,\kappa_2]s],\kappa_2]b \\
&= [[\kappa_2[\mathrm{id},\mathrm{id}]s,s],\kappa_2]b \\
&= [[\kappa_2,s],\kappa_2]b \\
&= [[[s,\kappa_2]\kappa_2,[s,\kappa_2]\kappa_1],\kappa_2]b \\
&= [[s,\kappa_2][\kappa_2,\kappa_1],[s,\kappa_2]\kappa_2]b \\
&= [s,\kappa_2][[\kappa_2,\kappa_1],\kappa_2]b = [s,\kappa_2]r'
\end{aligned}$$

Moreover:

$$
\begin{aligned}
(s \cdot r) \ovee (s \cdot r') &= (\nabla + 1)c \\
&= (\nabla + 1)[\mathrm{id}, \kappa_2](b + \mathrm{id})s \\
&= [(\nabla + 1)b, (\nabla + 1)\kappa_2]s \\
&= [r \ovee r', \kappa_2]s = s \cdot (r \ovee r')
\end{aligned}
$$

and

$$
\begin{aligned}
(r \cdot s) \ovee (r' \cdot s) &= (\nabla + 1)[[\kappa_1 + \mathrm{id}, \kappa_2 + \mathrm{id}], \kappa_2]((s + s) + \mathrm{id})b \\
&= (\nabla + 1)[(\nabla + 1)[\kappa_1 + \mathrm{id}, \kappa_2 + \mathrm{id}](s + s), (\nabla + 1)\kappa_2]b \\
&= [[(\nabla + 1)(\kappa_1 + \mathrm{id})s, (\nabla + 1)(\kappa_2 + \mathrm{id})s], \kappa_2]b \\
&= [[(\nabla\kappa_1 + \mathrm{id})s, (\nabla\kappa_2 + \mathrm{id})s], [s, \kappa_2]\kappa_2]b \\
&= [[(\mathrm{id} + \mathrm{id})s, (\mathrm{id} + \mathrm{id})s], [s, \kappa_2]\kappa_2]b \\
&= [[s, s], [s, \kappa_2]\kappa_2]b \\
&= [s[\mathrm{id}, \mathrm{id}], [s, \kappa_2]\kappa_2]b \\
&= [s, [s, \kappa_2]\kappa_2](\nabla + 1)b \\
&= [[s, \kappa_2]\kappa_1, [s, \kappa_2]\kappa_2](r \ovee r') \\
&= [s, \kappa_2](r \ovee r') = (r \ovee r') \cdot s
\end{aligned}
$$

■

Corollary E.7 In $\mathbb{B}$, predicates $\mathrm{Pred}(X) := \mathrm{Hom}_{\mathbb{B}}(X, 2)$ form an effect module over $M_{\mathbb{B}}$.

Proof
Similarly to Theorem E.6, we define, for all $p: X \to 2$, $r: 1 \to 2$:

$$
p \cdot r \;:=\; X \xrightarrow{p} 2 \xrightarrow{[r, \kappa_2]} 2
$$

It comes that $p \cdot 1 = [\kappa_1, \kappa_2]\, p = p$, $(p \cdot r) \cdot s = [s, \kappa_2][r, \kappa_2]\, p = [\overbrace{[s, \kappa_2]r}^{:= r \cdot s}, \overbrace{[s, \kappa_2]\kappa_2}^{= \kappa_2}]\, p = p \cdot (r \cdot s)$, and $\cdot$ being a bihomomorphism of effect algebras is shown in the exact same way as in the proof of Theorem E.6. ■

Lemma E.8 In $\mathbb{B}$, morphisms of the form $q: X \to n$ (called n-**tests**) are in one-to-one correspondence with predicates $p_1, \ldots, p_n : X \to 2$ summing to 1 (*i.e.* the p_i's are pairwise orthogonal and $p_1 \ovee \cdots \ovee p_n = 1$).

Proof
$\Longleftarrow$: If we have such predicates $p_1, \ldots, p_n$ with bound $b: X \to n + 1$, *i.e.*

$$
\begin{cases}
\forall 1 \leq i \leq n, \quad [\rhd_i, \kappa_1]\, b = p_i \\
(\nabla + \mathrm{id})\, b = 1 \;:=\; \kappa_1 !
\end{cases}
$$

then, by Lemma E.1:

$$\begin{tikzcd}
X \arrow[drr, bend left, "b"] \arrow[dr, dotted, "q"] \arrow[ddr, bend right, "!"'] & & \\
& n \arrow[r, "\kappa_1"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & n+1 \arrow[d, "\nabla+1 = !+\mathrm{id}"] \\
& 1 \arrow[r, "\kappa_1"'] & 1+1
\end{tikzcd}$$

The n-test $q\colon X \to n$ we obtain satisfies:

$$\rhd_i\, q = [\rhd_i, \kappa_2]\, \kappa_1 q = [\rhd_i, \kappa_2]\, b = p_i$$

It is uniquely determined, by joint monicity of $([\rhd_i, \kappa_2])_{1 \leq i \leq n}$ (Lemma E.3) and monicity of κ_1 (Lemma E.4).
$\Longrightarrow$: Conversely, given an n-test $q\colon X \to n$, by setting $p_i \;:= \rhd_i\, q = [\rhd_i, \kappa_2]\, \kappa_1 q\colon X \to 2$, it comes that $\kappa_1 q$ is a bound making the p_i's orthogonal, and

$$\bigovee_{i=1}^{n} p_i \quad := \quad (\nabla + \mathrm{id})\, \kappa_1 q = \kappa_1 \underbrace{\nabla q}_{=\,!\colon X \to 1} = 1$$

■

E.3 Convex sets over an effect monoid

Definition E.1 — Category Conv_M **of convex sets over an effect monoid** M

- **Objects**: Convex sets $X \in \mathrm{Conv}_M$ over of M are given by a carrier set X closed under finite convex combinations with coefficients in M: for all $x_1, \dots, x_n \in X$ and all $r_1, \dots, r_n \in M$ pairwise orthogonal and summing to 1, there exists $\sum\limits_{1 \leq i \leq n} r_i \,|x_i\rangle \in X$, where these convex combinations satisfy

$$1\,|x\rangle = x \qquad \text{and} \qquad \sum_i r_i \left| \sum_j s_{i,j}\,|x_{i,j}\rangle \right\rangle = \sum_{i,j} (r_i \cdot s_{i,j})\,|x_{i,j}\rangle$$

- **Morphisms** $f\colon X \to Y$ are function between the underlying sets preserving convex combination: $f\Big(\sum\limits_i r_i\,|x_i\rangle\Big) = \sum\limits_i r_i\,|f(x_i)\rangle$

More abstractly, convex sets over M are straighforwardly seen to coincide with algebras of the distribution monad $\mathcal{D}_M$ (and Jacobs even define them as such in [Jac15]):

Proposition E.3

$$\mathrm{Conv}_M \;\cong\; \mathcal{EM}(\mathcal{D}_M)$$

Lemma E.9 If M is an effect monoid, one has the adjunction:

$$\begin{tikzcd}
(\boldsymbol{\mathcal{EMod}}_M)^{\mathrm{op}} \arrow[rr, bend left, "\mathrm{Hom}_{\boldsymbol{\mathcal{EMod}}_M}(-,M)"] \arrow[rr, phantom, "\top"] & & \mathrm{Conv}_M \arrow[ll, bend left, "\mathrm{Hom}_{\mathrm{Conv}_M}(-,M)"]
\end{tikzcd}$$

Proof
Given a convex set $X \in \mathrm{Conv}_M$, we endow $\mathrm{Hom}_{\mathrm{Conv}_M}(X, M)$ with an effect monoid structure by defining $\perp$, $\oslash$ and $r \cdot (-)$ (for all $r \in M$) pointwise, which yields a contravariant functor $\mathrm{Hom}_{\mathrm{Conv}_M}(-, M)$ acting on morphisms by precomposition. In the other direction: if N is an effect module, $\mathrm{Hom}_{\mathbf{EMod}_M}(N, M)$ is a convex set, by setting $\sum_i r_i f_i \;:=\; x \mapsto \bigovee_i r_i \cdot f_i(x)$ (which remains a morphism of effect modules), and the action on morphisms, likewise, is given by precomposition.

On top of that, we have the natural bijection

$$\Phi_{N,X} \colon \begin{cases} \mathrm{Hom}_{\mathrm{Conv}_M}\left(X, \mathrm{Hom}_{\mathbf{EMod}_M}(N, M)\right) & \xrightarrow{\cong} \quad \overbrace{\mathrm{Hom}_{(\mathbf{EMod}_M)^{\mathrm{op}}}\left(\mathrm{Hom}_{\mathrm{Conv}_M}(X, M),\, N\right)}^{\mathrm{Hom}_{\mathbf{EMod}_M}(N, \mathrm{Hom}_{\mathrm{Conv}_M}(X,M))} \\ X \xrightarrow{\phi} \mathrm{Hom}(N, M) & \longmapsto \quad a \mapsto (x \mapsto \phi(x)(a)) \end{cases}$$

■

E.4 State and Predicate functors

Lemma E.10 — State functor. For all $X \in \mathbb{B}$, the set of states $\mathrm{Stat}(X) \;:=\; \mathrm{Hom}_{\mathbb{B}}(1, X)$ can be endowed with the structure of a convex set over the effect monoid of scalars $M_{\mathbb{B}}$. This yields a functor

$$\mathrm{Stat} \colon \mathbb{B} \longrightarrow \mathrm{Conv}_{M_{\mathbb{B}}}$$

Proof
Let $r_1, \dots, r_n \in M_{\mathbb{B}}$ be n pairwise orthogonal scalars summing to 1, and $\omega_1, \dots, \omega_n \in \mathrm{Stat}(X)$. By Lemma E.8, there exists a map $q \colon 1 \to n$ such that $\rhd_i\, q = r_i$ for all $1 \le i \le n$. We set

$$\sum_{i=1}^{n} r_i \,|\omega_i\rangle \;:=\; [\omega_1, \dots, \omega_n]\, q \colon 1 \to X$$

We check that we do have a convex set. If $r_i,\ s_{i,j} \colon 1 \to 2$ respectively correspond to $q \colon 1 \to n$ and $q_i \colon 1 \to m_i$ by Lemma E.8, and $\omega,\ \omega_{i,j} \colon 1 \to X$, for $1 \le i \le n,\ 1 \le j \le m_i$:

$$\begin{aligned} \kappa_1 \,|\omega\rangle &= [\omega]\, \mathrm{id} && \text{since } \rhd_1 \mathrm{id} = \kappa_1 \\ &= \omega \end{aligned}$$

and

$$\begin{aligned} \sum_i r_i \left| \sum_j s_{i,j}\, |\omega_{i,j}\rangle \right\rangle &= \sum_i r_i \,|[\omega_{i,1}, \dots, \omega_{i,m_i}]\, q_i\rangle \\ &= [[\omega_{1,1}, \dots, \omega_{1,m_1}]\, q_1, \dots, [\omega_{n,1}, \dots, \omega_{n,m_n}]\, q_n]\, q \\ &= [[\omega_{1,1}, \dots, \omega_{1,m_1}], \dots, [\omega_{n,1}, \dots, \omega_{n,m_n}]]\, (q_1 + \dots + q_n)\, q \end{aligned}$$

Note that, due to $r_i = \rhd_i\, q$ for all $1 \le i \le n$, it comes that for all $1 \le j \le m_i$:

$$\begin{aligned}
r_i \cdot s_{i,j} &= [s_{i,j}, \kappa_2]\, r_i \\
&= [s_{i,j}, \kappa_2][\kappa_2, \dots, \kappa_1, \dots, \kappa_2]\, q && \text{where } \kappa_1 \text{is in position } i,\ \kappa_2 \text{ everywhere else} \\
&= [[s_{i,j}, \kappa_2]\kappa_2, \dots, [s_{i,j}, \kappa_2]\kappa_1, \dots, [s_{i,j}, \kappa_2]\kappa_2]\, q \\
&= [\kappa_2, \dots, s_{i,j}, \dots, \kappa_2]\, q \\
&= [\kappa_2, \dots, \rhd_j\ q_i, \dots, \kappa_2]\, q \\
&= [\kappa_2\, !\, q_1, \dots, \rhd_j\ q_i, \dots, \kappa_2\, !\, q_n]\, q \\
&= [\underbrace{\kappa_2!}_{:\, m_1 \to 2}, \dots, \rhd_j, \dots, \underbrace{\kappa_2!}_{:\, m_n \to 2}](q_1 + \cdots + q_n)\, q
\end{aligned}$$

but, by denoting by N the sum $m_1 + \cdots + m_n$ (left-parenthesized as a coproduct of 1) and putting $k_{i,j} \ := \ m_1 + \cdots + m_{i-1} + j$, we have:

$$[\kappa_2!, \dots, \rhd_j, \dots, \kappa_2!] = [\rhd_{k_{i,j}}\ \alpha_1, \dots, \rhd_{k_{i,j}}\ \alpha_i, \dots, \rhd_{k_{i,j}}\ \alpha_n] = \rhd_{k_{i,j}}\ [\alpha_1, \dots, \alpha_n]$$

where $\alpha_l \colon m_l \to N$ maps m_l to the corresponding summand of the codomain, for $1 \leq l \leq n$, and $\rhd_{k_{i,j}} \colon N \to 2$. As a consequence:

$$r_i \cdot s_{i,j} = \rhd_{k_{i,j}}\ \underbrace{[\alpha_1, \dots, \alpha_n](q_1 + \cdots + q_n)\, q}_{\text{the unique } N\text{-test corresponding to the } r_i \cdot s_{i,j}\text{'s via Lemma E.8}}$$

Therefore

$$\begin{aligned}
\sum_i r_i \left| \sum_j s_{i,j} \left|\omega_{i,j}\right\rangle \right\rangle &= [[\omega_{1,1}, \dots, \omega_{1,m_1}], \dots, [\omega_{n,1}, \dots, \omega_{n,m_n}]]\, (q_1 + \cdots + q_n)\, q \\
&= \ [[\omega_{1,1}, \dots, \omega_{1,m_1}, \dots, \omega_{n,1}, \dots, \omega_{n,m_n}]\, \alpha_1, \\
&\qquad \dots, [\omega_{1,1}, \dots, \omega_{1,m_1}, \dots, \omega_{n,1}, \dots, \omega_{n,m_n}]\, \alpha_n]\, (q_1 + \cdots + q_n)\, q \\
&= [\omega_{1,1}, \dots, \omega_{1,m_1}, \dots, \omega_{n,1}, \dots, \omega_{n,m_n}]\, [\alpha_1, \dots, \alpha_n](q_1 + \cdots + q_n)\, q \\
&= \sum_{i,j} (r_i \cdot s_{i,j})\, \left|\omega_{i,j}\right\rangle
\end{aligned}$$

Finally, we easily check that postcomposition preserves convex sums, so that, for every $f \colon X \to Y$, $\mathrm{Stat}(f)$ is a morphism of convex spaces:

$$\begin{aligned}
\mathrm{Stat}(f)\Big(\sum_{i=1}^n r_i \left|\omega_i\right\rangle \Big) &= f\, [\omega_1, \dots, \omega_n]\, q \\
&= [f\, \omega_1, \dots, f\, \omega_n]\, q \\
&= \sum_{i=1}^n r_i \left|\mathrm{Stat}(f)(\omega_i)\right\rangle
\end{aligned}$$

■

Lemma E.11 — Predicate functor. For all $X \in \mathbb{B}$, the set of predicates $\mathrm{Pred}(X) \ := \ \mathrm{Hom}_{\mathbb{B}}\,(X, 2)$ forms an effect module over the effect monoid of scalars $M_{\mathbb{B}}$, yielding a functor

$$\mathrm{Pred} \colon \mathbb{B} \longrightarrow (\boldsymbol{\mathcal{EMod}}_{M_{\mathbb{B}}})^{\mathrm{op}}$$

Proof

By Corollary E.7, we already know that $\mathrm{Pred}(X)$ forms an effect module over $M_{\mathbb{B}}$. We thus require that $\mathrm{Pred}(f) \ := \ (-) \circ f \colon \ \mathrm{Pred}(Y) \to \mathrm{Pred}(X)$ be a map of effect modules, for every $f \colon X \to Y$. It is indeed

- *a map of effect algebras*: let $p, q\colon X \to 2$ be orthogonal predicates via a bound $b\colon X \to 3$. Then, clearly, bf is a bound for pf and qf. It comes that

$$\mathrm{Pred}(p) \ovee \mathrm{Pred}(q) = pf \ovee qf = (\nabla + \mathrm{id})bf = (p \ovee q)f = \mathrm{Pred}(p \ovee q)$$

- *a map preserving effect module structure*: let $p\colon X \to 2$ be a predicate, and $r\colon 1 \to 2$ a scalar. Then

$$\mathrm{Pred}(p \cdot r) = [r, \kappa_2]\, p\, f = \mathrm{Pred}(p) \cdot r$$

■

Definition E.2 — Logical validity If $X \in \mathbb{B}$, for every state $\omega\colon 1 \to X$ and predicate $p\colon X \to 2$, we define the **logical validity** as the scalar:

$$\omega \vDash p \ := \ p \circ \omega\colon 1 \to 2$$

This seemingly simplistic definition has a surprising large array of varied interpretations, depending on the effectus $\mathbb{B}$. We give a brief account of some of them, a more detailed description can be found in [Cho+15; Jac15]:

Examples in various effecti

Effectus $\mathbb{B}$	Predicates $X \xrightarrow{p} 2$	State $1 \xrightarrow{\omega} X$	Validity $\omega \vDash p$
Set	Subsets $p \subseteq X$	Elements $\omega \in X$	$\omega \in p$
$\mathcal{K}\ell(\mathcal{D}_M)$	Fuzzy predicate $p\colon X \to M$	Distributions/Convex sums $\omega = \sum\limits_{x \in X} \omega(x)\,\|x\rangle$	Expectation $\sum\limits_{x \in X} \omega(x) \cdot p(x)$
$\mathcal{K}\ell(\mathcal{G})$	Measurable maps $p\colon X \to [0,1]$	Probability measures $\omega \in \mathcal{G}(X)$	Continuous expectation $\int\limits_{x \in X} p \,\mathrm{d}\omega$
$\mathcal{DL}^{\mathrm{op}}$	Elements $p \in X$ that have a complement	Prime filters $\omega \subseteq X$	$p \in \omega$
$\mathcal{BA}^{\mathrm{op}}$	Elements $p \in X$	Ultrafilters $\omega \subseteq X$	$p \in \omega$
$\mathbf{Rng}^{\mathrm{op}}$	Idempotents $p \in X$	$\mathbb{Z}$-point $\omega\colon X \to \mathbb{Z}$	$\omega(p) \in \{0, 1\}$
${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$	$p \in X;\, 0 \le p \le 1$	$\omega\colon X \to \mathbb{C}$	$0 \le \omega(p) \le 1$
${\mathbf{C}^*_{\mathrm{PU}}}^{\mathrm{op}}$ when $X := \mathcal{B}(\mathscr{H})$ with $\mathscr{H}$ finite-dimensional	Effects $p\colon \mathscr{H} \to \mathscr{H};\, 0 \le p \le \mathrm{id}$	density matrix $\omega \in \mathcal{DM}(\mathscr{H})$	$\mathrm{Tr}(\omega p)$

where

- $\mathcal{G}$ is the continuous distribution (Giry) monad
- $\mathcal{BA}$ (resp. $\mathcal{DL}$) is the category of Boolean algebras (resp. distributive lattices)
- **Rng** is the category of rings
- $\mathbf{C}^*_{\mathrm{PU}}$ is the category of C^*-algebras and positive unital maps
- if $\mathscr{H}$ is a complex Hilbert space, $\mathcal{B}(\mathscr{H})$ denotes the continuous linear operators on $\mathscr{H}$ and if $\mathscr{H}$ is finite-dimensional, $\mathcal{DM}(\mathscr{H})$ are its density matrices (positive operators whose trace equals one)

F. C^*-algebras

Definition F.1 — Continuous and bounded linear maps Let $f\colon X \to Y$ be a linear map (sometimes called *operator*, when X and Y are thought as vector spaces of maps) between two normed spaces $(X, \| - \|_X)$ and $(Y, \| - \|_Y)$.

- f is **continuous** if, for every $x \in X$ and every sequence (x_n), if $x_n \longrightarrow x$ (*i.e.* $\lim_{n\to\infty} \|x_n - x\|_X = 0$) then $f(x_n) \longrightarrow f(x)$
- f is **bounded** if there exists a constant scalar λ such that $\|f(x)\|_Y \leq \lambda \|x\|_X$ for all $x \in X$. The smallest such λ is denoted by $\|f\|$ and is called the **operator norm** of f.

It is a well-known fact that a linear map between normed spaces is continuous iff it is bounded. Bounded linear maps of (operator) norm lower or equal to 1 are referred to as **short maps**.

Definition F.2 — Banach spaces and Banach algebras Recall that **Banach spaces** are normed vector spaces that are complete in the metric induced by the norm. They form a (closed monoidal) category $\mathbf{Ban}_{\mathbf{short}}$, with short linear maps as morphisms. If we require the linear operators be only bounded, the resulting category is denoted by **Ban**.
A (associative unital) **Banach algebras** is a monoid in $\mathbf{Ban}_{\mathbf{short}}$, *i.e.* a Banach space endowed with a bilinear associative multiplication satisfying

$$\|x \cdot y\| \leq \|x\| \, \|y\|$$

Banach spaces whose norm comes from a complex inner product are of paramount importance in a large array of subfields of mathematics and physics [DM05]: ranging from thermodynamics and ergodic theory to signal processing, Fourier analysis and, more importantly for us, quantum mechanics. These are called *Hilbert spaces*, a name coined by Von Neumann in 1927, even though Hilbert was not the first in the know, as evidenced by his asking ≪ Dr. Von Neumann, I would like to know what is a Hilbert space? ≫, bewildered, at the end of a lecture given by Von Neumann in Göttingen in 1929 (see [Dur+88, p. 330]).

Definition F.3 A (complex) **Hilbert space** $\mathscr{H}$ is

- a complex inner product vector space, *i.e.* a complex vector space equipped with an inner product $\langle - \mid \cdot \rangle\colon H \times H \to \mathbb{C}$ (also written $\langle -, \cdot \rangle$) which is conjugate symmetric ($\langle x \mid y \rangle = \overline{\langle y \mid x \rangle}$) positive ($\langle x, x \rangle \geq 0$) definite ($\langle x \mid x \rangle = 0 \implies x = 0$) and linear in its second[a] component
- that is a Banach space for the norm $\| - \| := \sqrt{\langle - \mid - \rangle}$ induced by the inner product.

We denote by **Hilb** (resp. $\mathbf{Hilb}_{\mathbf{short}}$) the subcategory of **Ban** (resp. $\mathbf{Ban}_{\mathbf{short}}$) comprised of Hilbert spaces and bounded linear maps.

[a] physics convention, to be compatible with Dirac's bra-ket notation

Definition F.4 A $*$**-ring** is a ring R together with an **anti-involution**, *i.e.* an involution $(-)^*\colon R \to R$ which is an *antimorphism* of rings: it preserves addition and 1, but reverses the order of multiplication ($(xy)^* = y^* x^*$).

In a $*$-ring, fixed points of $(-)^*$ are called **self-adjoint** elements.

Definition F.5 A $*$**-algebra** is a $*$-ring $(A, (-)^*)$ which is a unital associative algebra (*i.e.* a monoid in the category of modules over the ring of scalars) over a commutative $*$-ring $(R, \overline{(-)})$ such that $(-)^*$ is *antilinear* ($(rx)^* = \overline{r} x^*$).
With $*$-homomorphisms (algebra homomorphisms preserving $(-)^*$), $*$-algebras form a category. A **sub-$*$-algebra** of a $*$-algebra A is a complex linear subspace of A containing the unit $1 \in A$ and closed under multiplication and $(-)^*$.

Example F.1 — Bounded operators A key example of $*$-algebra is the set $\mathcal{B}(\mathscr{H}) \subseteq \mathrm{Hom}_{\mathbf{Hilb}}(\mathscr{H}, \mathscr{H})$ of **bounded operators** on a Hilbert space. The complex vector space structure is defined pointwise, multiplication is composition, and anti-involution is given by taking adjoint operators: if $T \in \mathcal{B}(\mathscr{H})$, the adjoint T^* is the unique operator such that $\langle T(x) \mid y\rangle = \langle x \mid T^*(y)\rangle$ for every $x, y \in \mathscr{H}$ (the existence and uniqueness of which stems from Riesz-Fréchet representation theorem).

Definition F.6 A C^***-algebra** is a Banach algebra A over $\mathbb{C}$ which is also a $*$-algebra $(A, (-)^*)$ over $(\mathbb{C}, \bar{(-)})$ satisfying the C^* *identity*:

$$\|x^* x\| = \|x\| \, \|x^*\|$$

In a C^*-algebra A, elements x that can be written $x = s^* s$ for some $s \in A$ are said to be **positive**. This endows A with a partial order given by $x \leq y \iff x - y$ is positive. A linear map between C^*-algebras is said to be **unital** if it preserves 1, **positive** if it sends positive elements to positive elements.

The previous definition of C^*-algebra is an abstraction of the original notion introduced by Irving Ezra Segal in 1947 [Seg47] in the following more concrete form, nowadays referred to as C^**-algebra of operators*:

Example F.2 — C^*-algebra of operators A C^***-algebra of operators** on a Hilbert space $\mathscr{H}$ is a sub-$*$-algebra B of the Banach $*$-algebra of bounded linear operators $\mathcal{B}(\mathscr{H})$ (the norm being the operator one) which is closed in the norm topology (*i.e.* for every sequence $(x_n) \in B^{\mathbb{N}}$ and $x \in \mathcal{B}(\mathscr{H})$, if $\lim_{n\to\infty} \|x_n - x\| = 0$ then $x \in B$). It is the prototype of a C^*-algebra.

For more details about C^*-algebras, we refer the reader to [Arv98; Dix77; Sak98; TT03].

G. Effectuses v Effect Monoids/Tricocycloids

G.1 From an effectus to an effect monoid

Theorem G.1 — Scalars of an effectus with normalisation form an effect monoid with normalisation.. If $\mathbb{B}$ is an effectus with normalisation, $M_{\mathbb{B}} := \mathrm{Hom}_{\mathbb{B}}(1,2)$ is an effect monoid with normalisation. ■

Proof
By Theorem E.6, it is an effect monoid. We will show that is has normalisation. Let $r, s\colon 1 \to 2$ such that $\kappa_1 \neq r \perp s$, *i.e.* there exists a bound $b\colon 1 \to 3$ such that $\begin{cases} r = 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{X/}} 2 \\ s = 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{I/}} 2 \end{cases}$ (wlog, because the relation $\perp$ is symmetric). Let us show that

$$\exists! s'. \quad s = r^{\perp} s' := 1 \xrightarrow{r} 2 \xrightarrow{\times} 2 \xrightarrow{s'+1} 2+1 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2$$

- *Existence of* s': Set $s' := (\!| 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 |\!)$. Then

$$\underbrace{1 \xrightarrow{r} 2 \xrightarrow{\times} 2}_{1 \xrightarrow{b} 3 \xrightarrow{\mathrm{X/}} 2 \;=\; 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{!+1} 2} \xrightarrow{s'+1} 2+1 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2 \underset{\text{definition of } s'}{\underset{\uparrow}{=}} 1 \xrightarrow{b} \underbrace{3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2}_{3 \xrightarrow{\mathrm{I/}} 2} = s$$

- *Uniqueness of* s': Let s' be a scalar satisfying the normalisation property. Since

$$\underbrace{1 \xrightarrow{r} 2 \xrightarrow{\times} 2}_{1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{!+1} 2} \xrightarrow{s''+1} \underbrace{2+1 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2}_{3 \xrightarrow{\mathrm{I/}} 2} = s = 1 \xrightarrow{b} \underbrace{3 \xrightarrow{\mathrm{I/}} 2}_{3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{\mathrm{I/}} 2}$$

  and owing to the maps $\mathrm{I/}$ and $\mathrm{X/}$ being jointly monic in $\mathbb{B}$, it suffices to show that

$$1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{!+1} 2 \xrightarrow{s''+1} 3 \xrightarrow{\mathrm{X/}} 2 = 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{\mathrm{X/}} 2$$

  to conclude that $b; \mathrm{IX}; (!+1); (s''+1) = b; \mathrm{IX}$, hence $s'' = (\!| 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 2 |\!) = s'$ by uniqueness of normalisation in an effectus. And it is the case, because

$$1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{!+1} 2 \xrightarrow{s''+1} \underbrace{3 \xrightarrow{\mathrm{X/}} 2}_{3 \xrightarrow{\mathrm{XI}} 3 \xrightarrow{\mathrm{I/}} 2 \;=\; 3 \xrightarrow{\mathrm{XI}} 3 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2} = 1 \xrightarrow{r} 2 \xrightarrow{\times} 2 \xrightarrow{s''+1} 3 \xrightarrow{\mathrm{XI}} 3 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2$$

  but the right-hand side is the scalar $r^{\perp} s''^{\perp} \overset{\text{Proposition D.1}}{=} (r \oslash \overbrace{r^{\perp} s''}^{=s})^{\perp} = (r \oslash \overbrace{r^{\perp} s'}^{=s})^{\perp} = r^{\perp} s'^{\perp}$. Therefore

$$1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{!+1} 2 \xrightarrow{s''+1} 3 \xrightarrow{\mathrm{X/}} 2 = \underbrace{1 \xrightarrow{r} 2 \xrightarrow{\times} 2 \xrightarrow{s'+1} 3}_{1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3 \xrightarrow{!+1} 2 \xrightarrow{s'+1} 3 \;=\; 1 \xrightarrow{b} 3 \xrightarrow{\mathrm{IX}} 3} \overbrace{\xrightarrow{\mathrm{XI}} 3 \xrightarrow{\mathrm{II}} 1+2 \xrightarrow{1+!} 2}^{3 \xrightarrow{\mathrm{X/}} 2}$$

■

G.2 Effectus and Kleisli category of the distribution monad over the scalars

Definition G.1 — Pairwise orthogonality: n predicates $p_1, \dots, p_n \colon X \to 2$ are said to be pairwise orthogonal if there exists a common bound $b \colon X \to n+1$ such that $\forall 1 \leq i \leq n, \quad [\rhd_i, \kappa_2]\, b = p_i$. As it happens, their sum is defined as $\bigovee_{i=1}^{n} p_i \quad := \quad (\nabla + \mathrm{id})\, b$

NB Such a bound b is unique by joint monicity of the family $([\rhd_i, \kappa_2])_{1 \leq i \leq n}$ (Lemma E.3).

Proposition G.1 In $\mathbb{B}$, if n predicates $p_1, \dots, p_n \colon X \to 2$ are pairwise orthogonal: for all $s_1, \dots, s_n \colon 1 \to 2$, so are the predicates $p_1 \cdot s_1, \dots, p_n \cdot s_n \colon X \to 2$.

Proof
Let $b \colon X \to n+1$ be a bound for $p_1, \dots, p_n$. Then $c \; := \; [[\kappa_1 + 1, \dots, \kappa_n + 1], \kappa_2]\, (s_1 + \cdots + s_n + 1)\, b \colon X \to n+1$ is a bound for $p_1 \cdot s_1, \dots, p_n \cdot s_n$: for all $1 \leq i \leq n$

$$\begin{aligned}
[\rhd_i, \kappa_2]\, c &= [\rhd_i, \kappa_2][[\kappa_1 + 1, \dots, \kappa_n + 1], \kappa_2]\, (s_1 + \cdots + s_n + 1)\, b \\
&= [[\rhd_i, \kappa_2][\kappa_1 + 1, \dots, \kappa_n + 1], [\rhd_i, \kappa_2]\kappa_2]\, (s_1 + \cdots + s_n + 1)\, b \\
&= [[[\rhd_i, \kappa_2](\kappa_1 + 1), \dots, [\rhd_i, \kappa_2](\kappa_n + 1)], \kappa_2]\, (s_1 + \cdots + s_n + 1)\, b \\
&= [[[\rhd_i\, \kappa_1, \kappa_2]s_1, \dots, [\rhd_i\, \kappa_n, \kappa_2]s_n], \kappa_2]\, b \qquad \text{where } \rhd_i\, \kappa_j = \begin{cases} \kappa_2 & \text{if } i \neq j \\ \kappa_1 & \text{else} \end{cases} \\
&= [[[\kappa_2, \kappa_2]s_1, \dots, [\kappa_1, \kappa_2]s_i, \dots, [\kappa_2, \kappa_2]s_n], \kappa_2]\, b \\
&= [[\kappa_2\, \underbrace{[\mathrm{id}, \mathrm{id}]s_1}_{= \mathrm{id} \colon 1 \to 1}, \dots, s_i, \dots, \underbrace{[\mathrm{id}, \mathrm{id}]s_n}_{= \mathrm{id} \colon 1 \to 1}], \kappa_2]\, b \\
&= [[\kappa_2, \dots, s_i, \dots, \kappa_2], \kappa_1]\, b \\
&= [[[s_i, \kappa_2]\kappa_2, \dots, [s_i, \kappa_2]\kappa_1, \dots, [s_i, \kappa_2]\kappa_2], [s_i, \kappa_2]\kappa_2]\, b \\
&= [s_i, \kappa_2][\rhd_i, \kappa_2]\, b = [s_i, \kappa_2]\, p_i = p_i \cdot s_i
\end{aligned}$$

■

Definition G.2 — Distribution monad $\mathcal{D}_M$: The (discrete) distribution monad $\mathcal{D}_M \colon \mathbf{Set} \to \mathbf{Set}$ over an effect monoid M is given

- on objects X, by:

$$\mathcal{D}_M(X) \; := \; \left\{ \phi \colon X \to M \;\middle|\; \mathrm{supp}(\phi) \text{ finite and } \bigovee_{x \in X} \phi(x) = 1 \right\} = \left\{ \sum_{i=1}^{n} r_i \,|x_i\rangle \;\middle|\; x_i \in X,\, r_i \in M,\, \bigovee_i r_i = 1 \right\}$$

  where $\mathrm{supp}(\phi) \subseteq X$ is the support of ϕ (set of elements $x \in X$ such that $\phi(x) \neq 0$). Such '*mass*' maps ϕ can be regarded as formal convex sums $\sum_{x \in X} \phi(x)\,|x\rangle$, where Dirac's ket notation is nothing but syntactic sugar drawing a distinction between elements $x \in \mathrm{dom}\, \phi$ and their occurrences in the formal sum. By convention, in these formal convex sums, $r_1\,|x\rangle + r_2\,|x\rangle$ will be identified with $(r_1 \ovee r_2)x$.
- on morphisms $f \colon X \to Y$ by:

$$\mathcal{D}_M(f) \; := \; \begin{cases} \mathcal{D}_M(X) & \longrightarrow \mathcal{D}_M(Y) \\ \sum_{i=1}^{n} r_i\,|x_i\rangle & \longmapsto \sum_{i=1}^{n} r_i\,|f(x_i)\rangle \end{cases}$$

The unit $\eta_X \colon X \to \mathcal{D}_M(X)$ and multiplication $\mu_X \colon {\mathcal{D}_M}^2(X) \to \mathcal{D}_M(X)$ of the monad are defined as:

$$\eta_X(x) \;:=\; 1\,|x\rangle \qquad\qquad \mu_X\Big(\sum_{i=1}^{n} r_i\,|\phi_i\rangle\Big) \;:=\; \sum_{x\in X}\Big(\bigovee_{i=1}^{n} r_i\cdot\phi_i(x)\Big)\,|x\rangle$$

NB The sum $\bigovee_{i=1}^{n} r_i\cdot\phi_i(x)$ is well defined thanks to Proposition G.1

Theorem G.2 Let $\mathbb{B}$ be an effectus whose objects are finite coproducts of 1. Then

$$\mathbb{B} \;\cong\; \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})$$

Proof
In what follows, as $\mathrm{Hom}_{\mathbb{B}}\,(m,n) \;\cong\; \prod_{1\le i\le m}\mathrm{Hom}_{\mathbb{B}}\,(1,n)$ by universal property of the coproduct, we denote by $q^1,\dots,q^m \colon 1\to n$ the m morphisms associated by this isomorphism to a morphism $q = [q^1,\dots,q^m]\colon m\to n$.
We put

$$F \;:=\; \begin{cases} \mathbb{B} & \longrightarrow\; \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})\\ n & \longmapsto\; n \\ m \xrightarrow{q} n & \longmapsto\; \begin{cases} m & \longrightarrow\; \mathcal{D}_{M_{\mathbb{B}}}(n)\\ k & \longmapsto\; \displaystyle\sum_{1\le i\le n} \rhd_i\, q^k\,|i\rangle\end{cases}\end{cases}$$

Let us show that F is a fully faithful functor, *i.e.* – as it is clearly bijective-on-objects – an isomorphism.

- F `is a functor:`
  - F `preserves identities`: for all $n\in\mathbb{B}$,

$$\begin{aligned} F(n \xrightarrow{\mathrm{id}} n) &= k \mapsto \sum_{1\le i\le n} \rhd_i\ \mathrm{id}^k\,|i\rangle && \text{where } \mathrm{id}^k = [\mathrm{id}^1,\dots,\mathrm{id}^n]\kappa_k = \mathrm{id}\,\kappa_k = \kappa_k\\ &= k \mapsto \sum_{1\le i\le n} \rhd_i\ \kappa_k\,|i\rangle && \text{where } \rhd_i\,\kappa_k = \begin{cases}\kappa_2 \;:=\; 0\in M_{\mathbb{B}} & \text{if } i\neq k\\ \kappa_1 \;:=\; 1\in M_{\mathbb{B}} & \text{else}\end{cases}\\ &= \underbrace{k\mapsto 1\,|k\rangle}_{=\eta_n} \;:=\; \mathrm{id}_n \in \mathrm{Hom}_{\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})}\,(n,n)\end{aligned}$$

  - F `preserves composition`: showing that

$$F(m \xrightarrow{q} n \xrightarrow{q'} n') \;:=\; k\mapsto \sum_{1\le j\le n'} \rhd_j\ (q'q)^k\,|j\rangle$$

  is equal to

$$\begin{aligned} F(m\xrightarrow{q} n);\, F(n\xrightarrow{q'} n') \;&:=\; m \xrightarrow{Fq} \mathcal{D}_{M_{\mathbb{B}}}(n) \xrightarrow{\mathcal{D}_{M_{\mathbb{B}}}(Fq')} {\mathcal{D}_{M_{\mathbb{B}}}}^2(n') \xrightarrow{\mu_{n'}} \mathcal{D}_{M_{\mathbb{B}}}(n')\\ &= k\mapsto \sum_{1\le i\le n}\rhd_i\,q^k\,|i\rangle \mapsto \sum_{1\le i\le n}\rhd_i\,q^k\,\underbrace{|F(q')i\rangle}_{|\sum_{1\le j\le n'}\rhd_j q'^i|j\rangle\rangle} \mapsto \sum_{j=1}^{n'}\Big(\bigovee_{i=1}^{n}(\rhd_i\,q^k)\cdot(\rhd_j\,q'^i)\Big)\,|j\rangle\end{aligned}$$

  amounts to showing that, for all $k\in[\![1,m]\!]$, $j\in[\![1,n']\!]$: $\rhd_j\ (q'q)^k = \bigovee_{i=1}^{n}(\rhd_i\,q^k)\cdot(\rhd_j\,q'^i)$.

This is indeed true, since:

$$\begin{aligned}
\bigovee_{i=1}^{n} (\rhd_i\, q^k) \cdot (\rhd_j\, q'^i) &= (\nabla + 1)\, \overbrace{[[\kappa_1 + 1, \ldots, \kappa_n + 1], \kappa_2]\Big(\sum_{1 \le i \le n} \rhd_j\, q'^i + 1\Big) \underbrace{\kappa_1 q^k}_{\text{bound for } \rhd_1 q^k, \ldots, \rhd_n q^k \text{ by E.8}}}^{\text{bound for } (\rhd_1 q^k)\cdot(\rhd_j q'^1), \ldots, (\rhd_n q^k)\cdot(\rhd_j q'^n) \text{ by Proposition G.1}} \\
&= [[(\nabla + 1)(\kappa_1 + 1), \ldots, (\nabla + 1)(\kappa_n + 1)], (\nabla + 1)\kappa_2]\Big(\sum_{1 \le i \le n} \rhd_j\, q'^i + 1\Big)\kappa_1 q^k \\
&= [[\underbrace{\nabla\kappa_1}_{=\mathrm{id}} + 1, \ldots, \underbrace{\nabla\kappa_n}_{=\mathrm{id}} + 1], \kappa_2]\kappa_1\Big(\sum_{1 \le i \le n} \rhd_j\, q'^i\Big) q^k \\
&= [\mathrm{id}, \ldots, \mathrm{id}]\Big(\sum_{1 \le i \le n} \rhd_j\, q'^i\Big) q^k \\
&= [\rhd_j\, q'^1, \ldots, \rhd_j\, q'^n] q^k = \rhd_j\, \overbrace{[q'^1, \ldots, q'^n]}^{=q'} q^k \\
&= \rhd_j\, ([q' q^1, \ldots, q' q^m])^k = \rhd_j\, (q' q)^k
\end{aligned}$$

- F `is faithful`: if $q, u\colon m \to n$ and $k \mapsto \sum_{1 \le i \le n} \rhd_i\, q^k\, |i\rangle = Fq = Fu = k \mapsto \sum_{1 \le i \le n} \rhd_i\, u^k\, |i\rangle$, then

$$\forall k, i. \quad [\rhd_i, \kappa_2]\kappa_1 q^k = \rhd_i\, q^k = \rhd_i\, u^k = [\rhd_i, \kappa_2]\kappa_1 u^k$$

  which results in $q^k = u^k$ for all k, by joint monicity of $([\rhd_i, \kappa_2])_{1 \le i \le n}$ (Lemma E.3) and monicity of κ_1 (Lemma E.4), whence $q = u$.
- F `is full`: for all $\phi \in \mathrm{Hom}_{\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}_{M_{\mathbb{B}}})}(m, n)$, ϕ is of the form

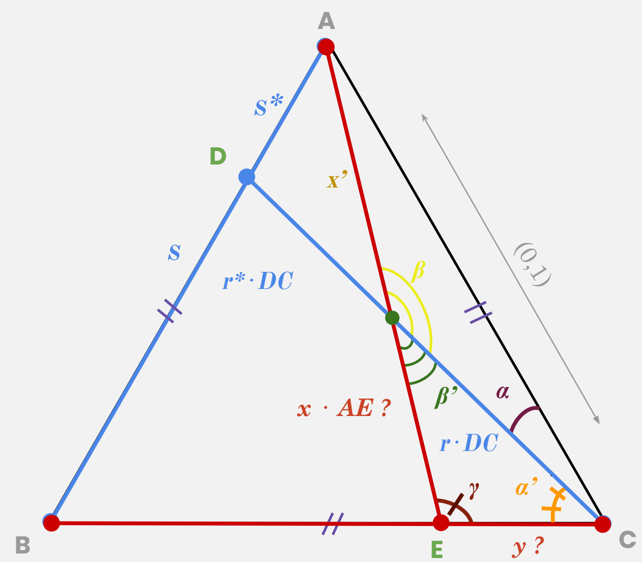
$$\phi\colon \begin{cases} m & \longrightarrow \ \mathcal{D}_{M_{\mathbb{B}}}(n) \\ k & \longmapsto \ \displaystyle\sum_{1 \le i \le n} r_{k,i}\, |i\rangle \end{cases}$$

  where the $r_{k,i}$'s are pairwise orthogonal. By Lemma E.8, for all $1 \le k \le m$, there exists a map $q_k\colon 1 \to n$ such that $\rhd_i\, q_k = r_{k,i}$ for all $1 \le i \le n$. We set $q := [q_1, \ldots, q_m]\colon m \to n$, so that $q^k = q_k$ for all $1 \le k \le m$. Consequently, $\phi = Fq$.

■

H. Geometric Interpretation

$(0,1)$ **being a symmetric tricocoycloid in** Set**: geometric interpretation**



Proof
In the symmetric tricocycloid $(0,1) \in \mathbf{Set}$,

$$v\colon \begin{cases} (0,1)^2 & \longrightarrow (0,1)^2 \\ (r,s) & \longmapsto (r \cdot s, r \diamond s) = (rs, \frac{rs^*}{(rs)^*}) \end{cases}$$

where $(-)^* \ := \ 1-(-) = (-)^\perp = \gamma$, turns the formal convex combination

$$r\big(s\,|A\rangle + s^*\,|B\rangle\,\big) + r^*\,|C\rangle$$

into the convex combination

$$rs\,|A\rangle + (rs)^*\left(\frac{rs^*}{(rs)^*}\,|B\rangle + \frac{r^*}{(rs)^*}\,|C\rangle\,\right) = r\cdot s\,|A\rangle + (r\cdot s)^\perp\big((r\diamond s)\,|B\rangle + (r\diamond s)^\perp\,|C\rangle\,\big)$$

This can be shown geometrically: in the above drawing, where ABC is equilateral (each side of which may be thought of as a copy of $H=(0,1)$), v turns (r,s) into (x,y).
Indeed, first note that:

$$DC = \sqrt{s+(1-s)^2} \qquad\qquad x' = \sqrt{(rDC)^2 + 1 - 2rDC\sqrt{1-\zeta^2}}$$

$$\zeta \ := \ \sin\alpha = \frac{\sqrt{3}(1-s)}{2\underbrace{DC}_{=\sqrt{s+(1-s)^2}}} \qquad\qquad \zeta^2 = \frac{3(1-s)^2}{4(s+(1-s)^2)}$$

Therefore, we first want to show that

$$y = \frac{2\,r\cdot DC}{\sqrt{1-\zeta^2}+\sqrt{3}\zeta - (\sqrt{3(1-\zeta^2)}-\zeta)\dfrac{(rDC)^2)+x'^2-1}{2x'rDC\zeta/x'}} \stackrel{?}{=} \frac{r(1-s)}{1-rs}$$

$$y/r = \frac{2DC}{(\sqrt{1-\zeta^2}+\sqrt{3}\cdot\zeta) - (\sqrt{3(1-\zeta^2)}-\zeta)\frac{\overbrace{((rDC)^2+x'^2-1)}^{2(rDC)^2-2rDC\sqrt{1-\zeta^2}}}{2r\cdot DC\cdot\zeta}}$$
$$= \frac{\overbrace{2\zeta(DC)}^{=\sqrt{3}(1-s)}}{\zeta(\sqrt{1-\zeta^2}+\sqrt{3}\zeta) - (\sqrt{3(1-\zeta^2)}-\zeta)(rDC-\sqrt{1-\zeta^2})}$$

But the denominator in the last expression is equal to

$$\zeta\sqrt{1-\zeta^2}+\sqrt{3}\zeta^2 - rDC\sqrt{3(1-\zeta^2)}+\frac{\sqrt{3}}{2}(1-s)r+\sqrt{3}(1-\zeta^2)-\zeta\sqrt{1-\zeta^2}$$
$$= 1-rs+\frac{rs}{2}+\frac{r}{2}-rDC\sqrt{1-\zeta^2} \quad \textbf{(H.1)}$$

and we claim that

$$\frac{rs}{2}+\frac{r}{2}-rDC\sqrt{1-\zeta^2}=0$$

Indeed:

$$\begin{aligned} r(1+s) &= 2rDC\sqrt{1-\zeta^2} \\ \iff \quad 1+2s+s^2 &= \underbrace{4(DC)^2(1-\zeta^2)}_{=4(DC)^2-3(1-s)^2} \\ \iff \quad 1+2s+s^2 &= 4s+(1-s)^2 \end{aligned}$$

and the last equality is clearly true.
Secondly, let us show that $x = rs$.
Note that $\frac{AE}{\sqrt{3}/2} = \frac{1}{\sin\gamma}$ – whence $AE = \frac{\sqrt{3}/2}{\sin\gamma}$ – and $x\cdot AE = \dfrac{\sin\alpha'\, rDC}{\sin\gamma}$.
As a result:

$$\begin{aligned} x\cdot AE &\overset{?}{=} rs\cdot AE \\ \iff \quad \frac{\overbrace{\sin\alpha'}^{\frac{1}{2}\left(\sqrt{3(1-\zeta^2)}-\zeta\right)}\cdot r\cdot DC}{\sin\gamma} &= rs\cdot\frac{\sqrt{3}/2}{\sin\gamma} \\ \iff \quad (\sqrt{1-\zeta^2}-\frac{\zeta}{\sqrt{3}})DC &= s \\ \iff \quad DC\sqrt{1-\zeta^2}-\frac{(1-s)}{2} &= s \\ \iff \quad 2DC\sqrt{1-\zeta^2}-1 &= s \\ \iff \quad \underbrace{4(DC)^2(1-\zeta^2)}_{4(s+(1-s)^2)-3(1-s)^2=4s+(1-s)^2} &= (s+1)^2 \end{aligned}$$

■

I. Implementation

I.1 Why3 proof of the “Tricocyloid to Effect Monoid” direction

```
module SymmetricTricocycloid

  type t
  function (++) t t : t
  function ( *) t t : t
  function (#_) t : t
  function v_inv t t : (t, t)

  clone export algebra.Assoc with type t = t, function op = ( *), axiom Assoc

  axiom Involution : forall r: t. #(#r) = r
  axiom Inverse_1 : forall r s: t. v_inv (r*s) (r ++s) = (r, s)
  axiom Inverse_2 : forall r s: t. let r', s' = v_inv r s in r'*s' = r && r'++s' = s

  axiom Symmetry_1 : forall r s: t. #(r*s) * (r ++ s) = r * #s
  axiom Symmetry_2 : forall r s: t. #(r*s) ++ (r ++ s) = #(r ++ #s)

  lemma Orthogonal_Symmetry_1: forall r s:t. #(r* s) * #(r ++ s) = #r
  lemma Orthogonal_Symmetry_2: forall r s:t. #(r* s) ++ #(r ++ s) = #s

  axiom Tricocycle_1 : forall r s t: t. (r ++ s*t) * (s ++  t) = r*s ++ t
  axiom Tricocycle_2 : forall r s t: t. (r ++ s*t) ++ (s ++  t) = r ++ s

  lemma Orthogonal_Tricocycle_1: forall r s t:t. (r ++ s*t) * #(s ++ t) = #(r*s ++ t) * (r ++ s)
  lemma Orthogonal_Tricocycle_2: forall r s t:t. (r ++ s*t) ++ #(s ++ t) = #(#(r*s ++ t) ++ (r ++ s))
end

module SymmetricTricocycloidLeftcancellative

  clone export SymmetricTricocycloid with axiom .
  axiom Left_cancellation : forall r s s': t. r*s = r*s' → s = s'
end

module SymmetricTricocycloidDoubleCancellation

  clone export SymmetricTricocycloidLeftcancellative with axiom .
  axiom Double_cancellation : forall r s r' s': t.
    r*s = s'*r' ∧ r * #s = #s' * r' → r = r'

end


module PCM

  type t
  constant zero : t
  function (+) t t : t
  predicate (˜) t t

  axiom Commutativity_sum : forall a b: t. a ˜ b
                                       → b ˜ a ∧ a + b = b + a
  axiom Associativity_sum : forall a b c: t. a ˜ b ∧ (a + b) ˜ c
                                       → b ˜ c ∧ a ˜ (b + c) ∧ (a + b) + c = a + (b + c)
  axiom Unit_law_sum : forall a: t. zero ˜ a ∧ zero + a = a
end


module PCM_endomorphism

  type t
  constant zero : t
  function (+) t t : t
  predicate (˜) t t

  function f t : t

  axiom Preservation_sum : forall a b: t. a ˜ b → f a ˜ f b ∧ f(a + b) = f a + f b
  axiom Preservation_zero : f zero = zero
end


module EffectAlgebra

  clone export PCM with axiom .

  function (#_) t : t
  constant one : t = #zero

  axiom Orthocomplement_existence : forall a: t. a ˜ #a ∧ a + #a = one
  axiom Orthocomplement_uniqueness : forall a b: t. a ˜ b ∧ a + b = one → b = #a

  axiom Orthocomplement_one_uniqueness : forall a: t. a ˜ one → a = zero
end


module EffectAlgebra_biendomorphism

  clone export EffectAlgebra with axiom .

  function g t t : t

  axiom Preservation_one : g one one = one

  (* homomorphism of PCM *)
  axiom Preservation_sum_1 : forall a a' b: t.
    a ˜ a' → g a b ˜ g a' b ∧ g (a + a') b = g a b + g a' b
  axiom Preservation_zero_1 : forall a: t. g zero a = zero

  axiom Preservation_sum_2 : forall a b b': t.
```

```
    b ~ b' → g a b ~ g a b' ∧ g a (b + b') = g a b + g a b'

  axiom Preservation_zero_2 : forall a: t. g a zero = zero

end


module EffectMonoid

  type t
  function ( *) t t : t

  clone export EffectAlgebra_biendomorphism with type t = t, function g = ( *), axiom .

  axiom Unit_law_left_mul : forall a: t. one * a = a
  axiom Unit_law_right_mul : forall a: t. a * one = a

  axiom Associativity_mul : forall a b c: t. a * (b * c) = (a * b) * c

end


module EffectMonoidNormalisation

  clone export EffectMonoid with axiom .

  axiom Normalisation : forall a b: t. a ≠ one ∧ a ~ b → exists c.  ( b = #a * c ∧ (forall d: t. b = #a * d → d = c))


  lemma Leftcancellation_nonzero : forall a b c: t. a ≠ zero ∧ a * b = a * c → b = c
end


theory Tricocycloid_to_EffectMonoidNormalisation

  use SymmetricTricocycloidDoubleCancellation

  type t' = Trico t | Zero | One

  let ghost function (+++) (a b: t') : t' =
    match a, b with
    | One, _                  → One
      (* keeping the function pure, but One +++ One may be considered undefined *)
    | Zero, _                 → Zero
    | _, One                  → Zero
    | r', Zero                → r'
    | Trico r, Trico s        → Trico (r ++ s)
    end

  let ghost function ( **) (a b: t') : t' =
    match a, b with
    | Zero, _ | _, Zero    → Zero
    | r', One | One, r'    → r'
    | Trico r, Trico s     → Trico (r*s)
    end

  let ghost function (#:_) (a: t') : t' =
    match a with
    | Zero     → One
    | One      → Zero
    | Trico r → Trico (#r)
    end

  (* [...] ELLIPSIS: CF. THE GITHUB REPO TO SEE THE FULL IMPLEMENTATION *)

  function norm t' t' : (t', t')

  axiom norm_0_x : forall x: t'. x ≠ Zero → norm Zero x = (x, Zero)
  axiom norm_x_0 : forall x: t'. norm x Zero = (x, One)
  axiom norm_rs : forall r s: t'. r ≠ Zero → norm (r ** s) (r ** #:s) = (r, s)

  predicate (~) (a b: t') =
    let r, s = norm a b in a = r ** s ∧ b = r ** #:s

  (* [...] ELLIPSIS: CF. THE GITHUB REPO TO SEE THE FULL IMPLEMENTATION *)

  clone export EffectMonoidNormalisation
        with type t = t',
              constant zero = Zero,
              function (+) = (+),
              predicate (~) = (~),
              function (#_) = (#:_),
              function ( *) = ( **),
              goal Commutativity_sum,
              goal Associativity_sum,
              goal Unit_law_sum,
              goal Orthocomplement_existence,
              goal Orthocomplement_uniqueness,
              goal Preservation_zero_1,
              goal Preservation_sum_2,
              goal Preservation_zero_2,
              goal Unit_law_left_mul,
              goal Unit_law_right_mul,
              goal Associativity_mul,
              goal Normalisation,
              goal Preservation_sum_1

end
```