Exercice - 70 - **CM 2, puis Mines (partiel)** 2012

Calculer

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

Cet exercice est signalé, par Vladimir Arnol'd, comme un exemple illustrant la décadence qui conduit de « lumières » à aujourd'hui. Il affirme que Barrow, Newton and Huygens would have solved it in a few minutes but present-day mathematicians are not, my opinion, capable of solving it quickly ! The only exception I know – Gerd Faltings – proves the rule.

Solution. On pose $\forall x \in]0,1],\ f(x) \overset{\text{déf}}{=} \frac{\sin(\tan x)-\tan(\sin x)}{\arcsin(\arctan x)-\arctan(\arcsin x)}$

On va utiliser les DL(0) à l'ordre [1] 7 de $\sin, \tan, \arcsin, \arctan$:

$$\begin{cases} \sin x & \underset{x\to 0}{=} x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{7!} + o(x^7) \\ \tan x & \underset{x\to 0}{=} x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + o(x^7) \\ \arcsin x & \underset{x\to 0}{=} x + \dfrac{x^3}{6} + \dfrac{3x^5}{40} + \dfrac{5x^7}{112} + o(x^7) \\ \arctan x & \underset{x\to 0}{=} x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + o(x^7) \end{cases}$$

$$f(x) \underset{x\to 0}{=} \frac{\tan x - \dfrac{\tan^3 x}{6} + \dfrac{\tan^5 x}{120} - \dfrac{\tan^7 x}{7!} - \sin x - \dfrac{\sin^3 x}{3} - \dfrac{2\sin^5 x}{15} - \dfrac{17\sin^7 x}{315} + o(x^7)}{\arctan x + \dfrac{\arctan^3 x}{6} + \dfrac{3\arctan^5 x}{40} + \dfrac{5\arctan^7 x}{112} - \arcsin x + \dfrac{\arcsin^3 x}{3} - \dfrac{\arcsin^5 x}{5} + \dfrac{\arcsin^7 x}{7} + o(x^7)}$$

$$\underset{x\to 0}{=} \frac{x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} - \frac{1}{6}\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right)^3 + \frac{1}{120}\left(x + \frac{x^3}{3}\right)^5 - \frac{x^7}{7!} - x + \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{7!} - \frac{1}{3}\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^3 - \frac{2}{15}\left(x - \frac{x^3}{6}\right)^5 - \frac{17x^7}{315} + o(x^7)}{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{1}{6}\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)^3 + \frac{3}{40}\left(x - \frac{x^3}{3}\right)^5 + \frac{5x^7}{112} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112} + \frac{1}{3}\left(x + \frac{x^3}{6} + \frac{3x^5}{40}\right)^3 - \frac{1}{5}\left(x + \frac{x^3}{6}\right)^5 + \frac{x^7}{7} + o(x^7)}$$

$$\underset{x\to 0}{=} \frac{(\frac{1}{3} - \frac{1}{6} + \frac{1}{6} - \frac{1}{3})x^3 + (\frac{2}{15} - \frac{1}{6} + \frac{1}{120} - \frac{1}{120} + \frac{1}{6} - \frac{2}{15})x^5 + (\frac{17}{315} - \frac{1}{2}(\frac{2}{15} + \frac{1}{9}) + \frac{1}{24\times 3} - \frac{1}{36} - \frac{1}{120} + \frac{1}{9} - \frac{17}{315})x^7 + o(x^7)}{(\frac{-1}{3} + \frac{1}{6} - \frac{1}{6} + \frac{1}{3})x^3 + (\frac{1}{5} - \frac{1}{6} + \frac{3}{40} - \frac{3}{40} + \frac{1}{6} - \frac{1}{5})x^5 + (\frac{-1}{7} + \frac{1}{2}(\frac{1}{9} + \frac{1}{5}) - \frac{1}{8} + \frac{1}{36} + \frac{3}{40} - \frac{1}{6} + \frac{1}{7})x^7 + o(x^7)}$$

$$\underset{x\to 0}{=} \frac{(\frac{1}{3} - \frac{1}{6} + \frac{1}{6} - \frac{1}{3})x^3 + (\frac{2}{15} - \frac{1}{6} + \frac{1}{120} - \frac{1}{120} + \frac{1}{6} - \frac{2}{15})x^5 + (\frac{17}{315} - \frac{1}{2}(\frac{2}{15} + \frac{1}{9}) + \frac{1}{24\times 3} - \frac{1}{36} - \frac{1}{120} + \frac{1}{9} - \frac{17}{315})x^7 + o(x^7)}{(\frac{-1}{3} + \frac{1}{6} - \frac{1}{6} + \frac{1}{3})x^3 + (\frac{1}{5} - \frac{1}{6} + \frac{3}{40} - \frac{3}{40} + \frac{1}{6} - \frac{1}{5})x^5 + (\frac{-1}{7} + \frac{1}{2}(\frac{1}{9} + \frac{1}{5}) - \frac{1}{8} + \frac{1}{36} + \frac{3}{40} - \frac{1}{6} + \frac{1}{7})x^7 + o(x^7)}$$

$$\underset{x\to 0}{=} \frac{\frac{-1}{30}x^7 + o(x^7)}{\frac{-1}{30}x^7 + o(x^7)} \underset{x\to 0}{\longrightarrow} 1$$

Donc

1. Retroussez vos manches !

$$\lim_{x\to 0}\frac{\sin(\tan x)-\tan(\sin x)}{\arcsin(\arctan x)-\arctan(\arcsin x)}=1$$

A propos de *the only exception*

Après l'effort, le réconfort ! Quid de l'affirmation d'ARNOL'D : "present-day mathematicians are not, my opinion, capable of solving it quickly ! The only exception I know – GERD FALTINGS – proves the rule"?
Apparemment, ARNOL'D avait posé cet exercice dans une conférence à Princeton, il y a une trentaine d'années. J'ai mis de côté ce commentaire croustillant d'un membre de mathoverflow, qui relate les faits :

"I heard this story from Dinesh Thakur several years ago. This is what he told me. When Arnold posed this question, Faltings immediately said 1. After the talk somebody complimented Faltings on his quickness, and Faltings replied that what immediately came to mind was that the answer had to be either 0 or 1. Since 1 was a more interesting answer, he went with that." [a]

a. Apprécions la modestie de Faltings !

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